臺灣大學數學系 103 學年度碩士班甄試試題

科目：線性代數

2013.10.18

1. (20%)
   (a) Let $A \in M_{m\times n}(F)$, $B \in M_{n\times p}(F)$ where F is a field.
   Show that $\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$.
   Moreover, if $n = p$ and B is invertible, show that $\text{rank}(AB) = \text{rank}(A)$.
   (b) Let $A \in M_{m\times n}(\mathbb{C})$. Show that $\text{rank}(A^*A) = \text{rank}(A)$ where A^* is the conjugate transpose of A.

2. (20%)
   (a) Let A be an $n \times n$ real symmetric matrix. Show that if λ is an eigenvalue of A in $\mathbb{C}$, then λ is real.
   (b) Let A be an $n \times n$ real symmetric matrix. Show that one can find an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

3. (20%)
   (a) Which $n \times n$ real matrices B have the property that $AB = BA$ for all $n \times n$ real matrices A? Justify your answer.
   (b) Let A, B be two $n \times n$ real symmetric matrices. Show that A and B are simultaneously diagonalizable if and only if $AB = BA$.

4. (20 %) For all $x \in \mathbb{R}^n$, we define the norm of x by $\|x\| = \sqrt{\langle x, x\rangle}$ where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^n$.
   Let $A = \begin{pmatrix} 0 & 4 & 0 & 4 \\ 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.
   (a) Find a vector p in the column space of A (the subspace of $\mathbb{R}^3$ spanned by the column vectors of A) such that $\|p - b\| \le \|A \cdot x - b\|$ for all $x \in \mathbb{R}^4$
   (b) Find $s \in \mathbb{R}^4$ such that $A \cdot s = p$ and s has the minimum norm, that is, $\|s\| \le \|v\|$ for all solutions v (in $\mathbb{R}^4$) of $A \cdot x = p$.
   (Justify your answers.)

5. (20 %) Find the Jordan form B of

$$A = \begin{pmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 5 & -3 \\ 4 & -1 & 3 & -1 \end{pmatrix}$$

   and the matrix P such that $B = P^{-1}AP$.
   (Notice! Show your works in details. No points will be assigned for non-substantial answers.)