臺灣大學數學系 102 學年度碩士班甄試試題
科目：線性代數

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1. [20%] Let V be an n-dimensional vector space over a field. Let $T : V \to V$ be a linear map. Show that the degree of the minimal polynomial of T equals

$$\max_{v \in V} \left\{ \dim\langle v, T(v), T^2(v), \cdots, T^{n-1}(v) \rangle \right\}.$$

(Here $\langle w_1, \cdots, w_r \rangle$ denotes the subspace spanned by $w_1, \cdots, w_r$.)

2. [20%] Consider the real $n \times n$ matrix $A = (a_{ij})$ satisfying

   - $a_{ij} \geq 0$ for all i, j,
   - $a_{ii} = 0$ for all i, and
   - $\sum_{j=1}^{n} a_{ij} = \gamma$ for all $i = 1, 2, \cdots, n$ for some constant $\gamma \neq 0$.

   Show that

   (a) If $\lambda \in \mathbb{R}$ is a real eigenvalue of A, then $-\gamma \leq \lambda \leq \gamma$.

   (b) γ is an eigenvalue of A and the corresponding eigenspace has dimension one.

   (c) The eigenspace corresponding to $-\gamma$ has dimension either zero or one.

3. [20%] Let $A = (a_{ij})$ be a real $n \times n$ symmetric matrix. Show that A is *positive definite* (meaning: $v^t A v > 0$ for any non-zero $v \in \mathbb{R}^n$ where v^t is the transport of v) if and only if, for any $r = 1, 2, \cdots, n$, we have

$$\det A_r > 0 \quad \text{where } A_r = (a_{ij})_{1 \leq i,j \leq r} \in M_r(\mathbb{R}).$$

4. [20%] Let $T : V \to W$ be a linear map between two finite dimensional vector spaces. Let V^* and W^* be the dual spaces of V and W, respectively. Prove that

   (a) T is injective if and only if the transport $T^* : W^* \to V^*$ is surjective.

   (b) T is surjective if and only if the transport $T^* : W^* \to V^*$ is injective.

   (Recall that the transport T^* is defined by $\big(T^*(f)\big)(v) = f\big(T(v)\big)$ for $f \in W^*, v \in V$.)

5. [20%] Let A be a real $n \times n$ matrix such that $A^t = -A$ (where A^t denotes the transport of A). Let $\lambda = a + bi$ be a complex eigenvalue of A where $a, b \in \mathbb{R}$ and $i^2 = -1$. Show that $a = 0$.