臺灣大學數學系
九十六學年度碩士班甄試試題
科目：微積分與線性代數

2006.11.3

Calculus and Linear Algebra

(1) (10%) The implicit function $z = z(x, y)$ satisfies

$$x^2 + y^2 + z^2 - xz - yz + 2x + 2y + 2z - 2 = 0.$$

Find the extreme values of z.

(2) (10%) Find the integral $\iint_{\mathbb{R}^2} e^{ax^2+bxy+cy^2}\,dxdy$ where $a < 0, b^2 < 4ac$.

(3) (20%) Test the convergence.
(a) $\sum_{n=1}^{\infty} \frac{1}{n}\ln(1+\frac{1}{n})$. (b) $\int_0^{\infty} \frac{\sin t}{\sqrt{t}}\,dt$.

(4) (10%) Solve the system of differential equations.

$$\begin{cases} \frac{dy}{dx} = 7y - z \\ \frac{dz}{dx} = 2y + 5z \end{cases}$$

(5) (20%) Let V be an n-dimensional real inner product space.
(a) Let T be a linear operator on V. Show that there is a nontrivial T-invariant subspace W such that $\dim W \leq 2$.
(b) Let T be an orthogonal operator on V. Show that T can be expressed as the composition of at most one reflection and at most $\frac{n}{2}$ rotations.

(6) (15%) Let $A = (a_{ij}) \in M_n(\mathbb{C})$. Define $\|A\| = \max_{i,j} |a_{ij}|$. Show that there is a matrix B arbitrary close to A such that B has n distinct eigenvalues.

(7) (15%) Let V be the vector space of polynomials with real coefficients and degree $\leq n$. Define the inner product $\langle f_1, f_2\rangle = \int_0^1 f_1(x)f_2(x)dx$. Let $M = \{x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n | a_i \in \mathbb{R}\} \subset V$. Find the distance from the origin to M.