臺灣大學數學系113學年度碩士班甄試筆試試題

科目：高等微積分　　2023.11.02

1. (15%) If every closed and bounded set of a metric space (M, d) is compact, does it follow that (M, d) is complete? If your answer is "yes", prove it; if your answer is "no", give a counter-example.

2. (20%) Determine the values of h for which the following series converges uniformly on $I_h = \{x \in \mathbb{R} : |x| \le h\}$:

$$\sum_{n=1}^{\infty} \frac{(n!)^2\, x^n}{(2n)!} \ .$$

Show your work.

3. (10%+10%+5%) Consider

$$(\star) \qquad F(x) = \int_0^\infty \frac{e^{-xt} - e^{-t}}{t}\, \mathrm{d}t$$

on $I = \{x \in \mathbb{R} : \frac{1}{2} \le x \le 2\}$.

   (a) Show that $(\star)$ converges on I, and $F(x)$ is continuous on I.

   (b) Show that

$$F'(x) = \int_0^\infty -e^{-xt}\, \mathrm{d}t \ .$$

   (c) Evaluate $F(x)$.

4. (5%+15%) Let f be a smooth function on $\mathbb{R}^n$ with $\det\big(\big[\frac{\partial^2 f}{\partial x_i \partial x_j}\big]_{1\le i,j\le n}\big) = 2$ everywhere.

   (a) Show that there exist an open neighborhood $U \subset \mathbb{R}^n$ of the origin and an open set $V \subset \mathbb{R}^n$ such that $\mathbf{x} = (x_1, \ldots, x_n) \mapsto Df(\mathbf{x}) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ is a bijection map from U and V, and its inverse is also a smooth map.

   (b) Denote the inverse map in part (a) by $\boldsymbol{\xi}(\mathbf{y}) = (\xi_1(\mathbf{y}), \ldots, \xi_n(\mathbf{y}))$. For any $\mathbf{y} \in V$, let

$$f^*(\mathbf{y}) = -f(\boldsymbol{\xi}(\mathbf{y})) + \sum_{i=1}^{n} y_i\, \xi_i(\mathbf{y}) \ .$$

   Compute $\det\big(\big[\frac{\partial^2 f^*}{\partial y_i \partial y_j}\big]_{1\le i,j\le n}\big)$.

5. (20%) Let $f(x)$ be a C^1 function for $x \in [0, \infty)$. Suppose that $f(x) \ge 0$ and $f'(x) \le 1$ for every $x \ge 0$, and $\int_0^\infty f(x)\, \mathrm{d}x$ converges. Does $\lim_{x\to\infty} f(x)$ exist? If your answer is "yes", determine the limit and prove it; if your answer is "no", give a counter-example.