臺灣大學數學系 110 學年度碩士班甄試試題

科目：高等微積分

2020.10.23

1. (10 points)
Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Assume that its Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$ has rank n everywhere and $f^{-1}(K)$ is compact whenever K is compact. Show that $f(\mathbb{R}^n) = \mathbb{R}^n$

2. (15 points)
Let $F(x, y, z) = (x, y, z)$, and S be the boundary of the region $x^2 + y^2 \leq z \leq \sqrt{2 - x^2 - y^2}$, oriented so that the normal points out of the region. Compute $\int\int_S F \cdot ndA$.

3. (20 points)
Let M be a metric space with countable elements. Prove or disprove that M is disconnected.

4. (20 points)
Let $f(x) = \frac{1}{4} + x - x^2$. For a real number x, define $x_{n+1} = f(x_n)$, where $x_0 = x$. (1) Given $x = 0$, show that the sequence $\{x_n\}$ converges and find its limit L. (2) Find all real numbers x such that their corresponding sequences all converge to L.

5. (20 points)
Let X consist of all real valued functions f on $[0, 1]$ such that
(1)$f(0) = 0$
(2)$||f|| = \sup\{\frac{|f(x)-f(y)|}{|x-y|^{1/3}}; x \neq y\}$ is finite.
Prove that $|| \cdot ||$ is a norm for X and X is complete with respect to this norm.

6. (15 points)
Assume $f : (a, b) \to \mathbb{R}$ is differentiable. Consider its derivative $f'(x)$, show that $f'(x)$ never has a jump discontinuity.