臺灣大學數學系 104 學年度碩士班甄試試題

科目：高等微積分

2014.10.24

1. (10%) Determine whether $\displaystyle\int_0^\infty x^{-\frac{1}{3}}e^{-x}\,\mathrm{d}x$ *converges* or not. Justify your answer.

2. (20% = 10% + 10%) For the following functions, determine whether they are *uniformly continuous* or not. Justify your answer.

   (a) $f(x) = \log(2014 + x^{10})$ for $x \in \mathbb{R}$, where log is the natural logarithm function.

   (b) For $(x, y) \in [-1, 1] \times [-1, 1]$,
   $$g(x,y) = \begin{cases} x\sin(\dfrac{y}{x}) & \text{when } x \neq 0\ , \\ 0 & \text{when } x = 0\ . \end{cases}$$

3. (20% = 10% + 10%) For the following sequences of continuous functions on $\mathbb{R}^1$, determine whether they have a *uniformly convergent* subsequence or not. Justify your answer.

   (a) $f_n(x) = \exp(-(x-n)^2)$, $n \in \mathbb{N}$.

   (b) $g_n(x) = \sin(x - n)$, $n \in \mathbb{N}$.

4. (20% = 10% + 10%) Consider the system of equations:
   $$x + yu + z^2 + x^3 = 0\ ,$$
   $$z + x^2 + z^2u - y^3 = 0\ .$$
   It is clear that $x = y = z = u = 0$ is a solution.

   (a) Near $(0,0,0,0)$, can the system be solved for x, z as continuously differentiable functions of $y, u \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$? Justify your answer.

   (b) Near $(0,0,0,0)$, can the system be solved for y, u as continuously differentiable functions of $x, z \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$? Justify your answer.

5. (20% = 10% + 10%) Suppose that $\{a_n\}_{n=1}^\infty$ is a sequence such that $\sum_{n=1}^\infty n|a_n|$ converges.

   (a) Show that both $\sum_{n=1}^\infty a_n\sin(nx)$ and $\sum_{n=1}^\infty na_n\cos(nx)$ *converges uniformly* for all $x \in \mathbb{R}$.

   (b) Let $f(x) = \sum_{n=1}^\infty a_n\sin(nx)$, and $g(x) = \sum_{n=1}^\infty na_n\cos(nx)$. Prove that the derivative of $f(x)$ is $g(x)$.

6. (10%) Let $f(x, y, z)$ be a smooth function on $\mathbb{R}^3$, and κ be a constant within $(0, 3)$. Prove that
   $$(3-\kappa)\iiint_{|\mathbf{x}|\le r} f^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \le r\iint_{|\mathbf{x}|=r} f^2\,\mathrm{d}S + \frac{1}{\kappa}\iiint_{|\mathbf{x}|\le r}|\mathbf{x}|^2|\nabla f|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
   for any $r > 0$. Here, $\mathbf{x} = (x, y, z)$ and ∇f is the gradient of f.

   Hint: Apply the divergence theorem and the Cauchy–Schwarz inequality.