1. Let $X_1, \cdots, X_n$ be a random sample from a distribution with c.d.f F and a continuous p.d.f. f. Given $0 < p < 1$, let η_p be the p-th quantile of F and $X_{(k)}$ be the k-th order statistic of the sample such that $k/n \to p$.

   (a) (10%) State the asymptotic distribution of $X_{(k)}$ and a set of sufficient conditions.

   (b) (20%) Proof the result in (a).

2. Let $X_1, \cdots, X_n$ be a random sample from a distribution with p.d.f.

$$f(x, \theta) = c(\theta)\{1 - e^{-|x|}\}\, I(|x| \leq \theta)\,,$$

   where $c(\theta)$ is a normalizing constant.

   (a) (10%) Find the maximum likelihood estimator of θ and denote it as $\hat{\theta}$.

   (b) (10%) Find the p.d.f of $\hat{\theta}$.

3. Let random variables $X_1, \cdots, X_n$, $n \geq 2$ be independent and identically distributed with density

$$f(x; \eta, \theta) = \theta^{-1} \exp\{-(x - \eta)/\theta\} I(\eta < x)\,,$$

   where $-\infty < \eta < \infty$ and $\theta > 0$ are both unknown.

   (a) (10 points) Find the maximum likelihood estimators of θ and η and denote them as $\hat{\theta}$ and $\hat{\eta}$.

   (b) (15 points) Find the distribution of $(n-1)(\hat{\eta} - \eta)/\hat{\theta}$.

4. Suppose random variable X has a Poisson distribution with mean μ. Assume a Gamma prior distribution Gamma(α, β) of μ, where α and β are known. Consider the loss function $l(\hat{\mu}, \mu) = (\hat{\mu} - \mu)^2/\mu$.

   (a) (10%) Find the Bayes estimator of μ.

   (b) (15%) Find a minimax estimator μ.