# 臺灣大學數學系

# 九十學年度第二學期碩博士班資格考試試題

# 統計與機率

1.

(20 points) The random variables $X_1, \dots, X_n$ are independent and $X_i$ is normally distributed with mean $\theta_i$ and variance $1$.

(11) Show that the most-powerful size $0.05$ test of $H_0 : \theta_i = 0,\ 1 \le i \le n$ versus $H_1 : \theta_i = 1/2,\ 1 \le i \le r$ and $\theta_i = -1/2,\ r+1 \le i \le n$ has critical region

$$\left\{ (x_1, \dots, x_n) : \sum_{i=1}^{r} x_i - \sum_{i=r+1}^{n} x_i > 1.645\sqrt{n} \right\}.$$

(12) How large must $n$ be to ensure that the power of this test is at least $0.9$?

2.

(20 points) Let $X_1, \dots, X_n$ be a random sample from the pdf

$$f(x|\theta) = \theta x^{-2}, \quad 0 < \theta \le x < \infty.$$

(21) (8 points) Find the maximum likelihood estimate (MLE) of θ.
(22) (12 points) Derive the asymptotic distribution the MLE of θ.

3.

(20 points) *Importance sampling* is a useful technique for calculating features of a distribution. Suppose we want to find $Eh(X)$ where the probability density function (pdf) of $Y_1, Y_2, \dots, Y_m$ is $f$ and $h$ is a function. But the pdf $f$ is difficult to simulate from. Instead, generate $Y_1, Y_2, \dots, Y_m$ from a pdf $g$ where the supports of $f$ and $g$ are the same and $Varh(X) < \infty$.

(31) Show that $E\left( m^{-1} \sum_{i=1}^{m} \frac{f(Y_i)}{g(Y_i)} h(Y_i) \right) = Eh(X)$.

(32) Show that $m^{-1} \sum_{i=1}^{m} \frac{f(Y_i)}{g(Y_i)} h(Y_i)$ converges to $Eh(X)$ in probability.

(33) Although the estimator of (31) has the correct expectation, in practice the estimator

$$\sum_{i=1}^{m} \left( \frac{f(Y_i)/g(Y_i)}{\sum_{j=1}^{m} f(Y_j)/g(Y_j)} \right) h(Y_i)$$

is preferred. Show that this estimator converges in probability to $Eh(X)$. Moreover, show that this estimator is superior to the one in (31).

4.

(20 points) Let $Y_1, Y_2, \ldots, Y_m$ and $Y$ be independent exponential random variables, with $f(x|\lambda) = \lambda^{-1} \exp(-x/\lambda)$, $x > 0$, and $f(y|\mu) = \mu^{-1} \exp(-y/\mu)$, $y > 0$. Due to data collection problem, we can observe $(Z_i, W_i)$ only, $1 \le i \le n$,with $Z_i = \min(X_i, Y_i)$. Here $X_i$ and $Y_i$ are assumed to be independent and identically distributed and

$$W_i = \begin{cases} 1 & \text{if } Z_i = X_i \\ 0 & \text{if } Z_i = Y_i \end{cases} .$$

Find the maximum likelihood estimate of λ.

5.

(20 points) Find the asymptotic distribution of $\hat{p}_n(1-\hat{p}_n)$ where $\hat{p}_n$ is the proportion of success of a binomial distribution with $n$ trials and the probability of success $p$. (Hint: You may consider $p = 1/2$ and $p \neq 1/2$ separately.)