臺灣大學數學系

九十學年度第一學期碩博士班資格考試試題

統計與機率

1

A random sample, $X_1, \ldots, X_n$, is drawn from a Pareto population with pdf

$$f(x|\theta, \tau) = \frac{\theta \tau^\theta}{x^{\theta+1}} I_{[\tau,\infty]}(x), \quad \theta > 0, \quad \tau > 0.$$

Here $I_{[\tau,\infty]}(x)$ is equal to 1 if $x \geq \tau$; 0, otherwise.

(11) Find the MLE of θ and τ.
Hint: (12) may help you to solve this problem.
(12) Show that the likelihood ratio test of

$$H_0 : \theta = 1, \tau \text{ unknown versus } H_1 : \theta \neq 1, \tau \text{ unknown},$$

has critical region of the form
$\{(x_1, \ldots, x_n) : T((x_1, \ldots, x_n)) \leq c_1 \text{ or } T((x_1, \ldots, x_n)) \geq c_2\}$, where
$0 < c_1 < c_2$ and

$$T = \log \left[\left(\prod_{i=1}^{n} X_i \right) / \left(\min_i X_i \right)^n \right].$$

2

Suppose that we have two independent random samples: $X_1, \ldots, X_{100}$ are exponential(θ), and $Y_1, \ldots, Y_{200}$ are exponential(μ). (i.e. The probability density function of X_1 is $\theta^{-1} \exp(-x/\theta)$.) Statistician A is asked to perform the test of

$$H_0 : \theta = \mu \text{ versus } H_1 : \theta \neq \mu.$$

Since he only have a standard normal table, he proposes the following 0.95 level test with critical region

$$\left| T - \frac{1}{3} \right| > \frac{1}{10} \frac{\sqrt{5}}{9\theta^{1/2}} z_{0.025}.$$

Here

$$T = \frac{\sum_{i=1}^{100} X_i}{\sum_{i=1}^{100} X_i + \sum_{i=1}^{200} Y_i}.$$

Do you think that it is a reasonable proposal? If your answer is YES, give a theoretical justification. Otherwise, propose an alternative and give a theoretical justification.

3

Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with one of two probability density functions. If $\theta = 0$ then

$$f(x|\theta) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases},$$

while if $\theta = 1$

$$f(x|\theta) = \begin{cases} 1/(2\sqrt{x}) & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Find the MLE of θ and show that it is consistent.

4

For two factors-starchy or sugary and green base leaf of white base leaf- the following counts for the progeny of self-fertilized heterozygotes were observed:

Type	Count
Starchy green	1997
Starchy white	906
Sugary green	904
Sugary white	32

According to genetic theory, the cell probabilities are $0.25(2+\theta)$, $0.25(1-\theta)$, $0.25(1-\theta)$, and 0.25θ, where θ ($0 < \theta < 1$) is a parameter related to the linkage of the factors.

(41) Find the mle of θ and its asymptotic variance.

(42) Form an approximate 95% confidence interval for θ based on part (41).

5

Let $X_1, \ldots, X_n$ be a random sample from a $N(0,1)$ population. Define

$$Y_1 = \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right|, \quad Y_2 = \frac{1}{n} \sum_{i=1}^{n} |X_i|.$$

Calculate EY_1 and EY_2.