臺灣大學數學系

八十九學年度第二學期碩博士班資格考試試題

機率

1

(16 points). Let $X_1, X_2, \ldots$ be uncorrelated random variables with $X_n = m \quad \forall n$ and $Var(X_n) = O(n^\delta), \quad 0 < \delta < 1$. Prove that the weak law of large numbers holds for $X_1, X_2, \ldots$.

2

(16 points). Let $X_1, X_2, \ldots$ be independent with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$. Prove that (i) $X_n \to 0$ in probability if and only if $p_n \to 0$ (ii) $X_n \to 0$ a.s. if and only if $\sum p_n < \infty$.

3

(18 points). (i) Prove that $X_n \Rightarrow c$ if only if $X_n \to c$ in probability, where c is a constant. (ii) Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be independent. Suppose $X_n \Rightarrow X$ and $Y_n \Rightarrow Y$, prove that $X_n + Y_n \Rightarrow X + Y$.

4

(16 points). Let X be of standard normal distribution. Use characteristic function to compute the moments of X, that is , $EX^k, \quad k = 1, 2, 3, 4, \ldots$.

5

(18 points). Define the ``upcrossing number" U_n of $X_0, X_1, X_2, \ldots$ over an interval [a,b], and prove the inequality

$$(b-a)EU_n \le E(X_n - a)^+ - E(X_0 - a)^+$$

for submartingale $X_0, X_1, X_2, \ldots$.

6

(16 points). Let $X_1, X_2, \ldots$ be a martingale. Prove that there exists an X_∞ such that $X_1, X_2, .., X_\infty$ forms a martingale if and only if $X_1, X_2, \ldots$ are uniformly integrable.