臺灣大學數學系

八十九學年度第一學期碩博士班資格考試試題

統計與機率

§機率與統計擇一科作答, 如兩科都作答, 皆以零分計算.
§機率 Probability (70/100)

1.
(15/100)(1.1) State and prove the weak law of large numbers.
(1.2) (*Weierstrass Approximation Theorem.*) Given a continuous function $f:[0,1]\to\mathbb{R}$ construct first the *Bernstein polynomials* $\{B_n(x;f), n\in\mathbb{N}\}$ associate with f. Then apply the weak law of large numbers to prove that $B_n(x;f)\to f(x)$ uniformly on $[0,1]$
.

2.
(20/100) (2.1) Show that for any random variables X one has

$$\sum_{n=1}^{\infty} P(|X|\geq n)\leq E(X)\leq 1+\sum_{n=1}^{\infty} P(|X|\geq n).$$

(2.2) Let (Y_n) be an iid sequence of random variables. Show that $E(|Y_1|)<\infty$ if and only if $\lim_{n\to\infty}|Y_n|/n=0$ almost surely (a.s.). (Hint. Apply Borel-Cantelli lemma and the result in (2.1) with $X=Y_1/\varepsilon$ for $\varepsilon>0.$)

3.
(20/100) (3.1) Let (X_n) be a sequence of random variables. Assume it is known that $\sum_{n=1}^{\infty} X_n$ converges in probability (i.p.) if and only if $\sum_{n=1}^{\infty} X_n$ converges a.s. Apply this result to prove that if $\sum_{n=1}^{\infty}\mathrm{Var}(X_n)<\infty$, then $\sum_{n=1}^{\infty}(X_n-E(X_n))$ converges a.s.
(3.2) Let (Z_n) be an iid sequence of random variables with $E(Z_n)=0$, Var $(Z_n)=1$. Denote $S_n=\sum_{j=1}^{n} Z_j$. Apply the result in (3.1) and Kronecker's lemma to find the range

of δ, as large as you can, for which $\frac{S_n}{n^{\frac{1}{2}}(\log n)^{\delta}} \longrightarrow 0$ a.s.

4.

(15/100) **(Solve only one of the following two questions.)**

(4.1)

Suppose $X_n, n \geq 0; Y_n, n \geq 1$ are random variables such that X_n converges to X_0 in distribution and Y_n converges to 0 i.p. Show that $X_n + Y_n$ converges to X_0 in distribution.

(4.2)

Let probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and sub-field $\mathcal{G} \subset \mathcal{F}$ be given. Let X, Y be two random variables such that $X, XY \in L_1(\Omega, \mathcal{F}, \mathcal{P})$ and Y is $\mathcal{G}$-measurable. Show that $E[XY|\mathcal{G}] = YE[X\mathcal{G}]$ a.s.

§統計

Let $\chi^2_{\alpha,df}$ denote the α-th quantile of Chi-square distribution with degree of freedom df.

df	21	22	23	24	25	26	27	28	29	30
$\chi^2_{0.05,df}$	11.59	12.34	13.09	13.85	14.61	15.38	16.15	16.93	17.71	18.49
$\chi^2_{0.95,df}$	32.67	33.92	35.17	36.42	37.65	38.89	40.11	41.34	42.56	43.77
df	31	32	33	34	35	36	37	38	39	40
$\chi^2_{0.05,df}$	19.28	20.07	20.87	21.66	22.47	23.27	24.07	24.88	25.7	26.51
$\chi^2_{0.95,df}$	44.99	46.19	47.4	48.6	49.8	51	52.19	53.38	54.57	55.76
df	41	42	43	44	45	46	47	48	49	50
$\chi^2_{0.05,df}$	27.33	28.14	28.96	29.79	30.61	31.44	32.27	33.1	33.93	34.76
$\chi^2_{0.95,df}$	56.94	58.12	59.3	60.48	61.66	62.83	64	65.17	66.34	67.5
df	51	52	53	54	55	56	57	58	59	60
$\chi^2_{0.05,df}$	35.6	36.44	37.28	38.12	38.96	39.8	40.65	41.49	42.34	43.19
$\chi^2_{0.95,df}$	68.67	69.83	70.99	72.15	73.31	74.47	75.62	76.78	77.93	79.08

1.

1. (A bio-assay problem) Suppose that the probability of death $p(x)$ is related to the

dose x of a certain drug in the following manner

$$p(x) = \frac{1}{1+e^{-(\alpha+\beta x)}},$$

where $\alpha > 0, \beta \in R$ are unknown parameters. In an experiment, k different(given) doses $x_1, x_2, \ldots, x_k$ of the drug are considered, dose level x_i is applied to n_i (given) animals and the number Y_i of deaths among them are recorded. Derive sufficient statistics for (α, β). (8 points)

2. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with p.d.f. $f(\cdot;\theta), \theta \in \Omega \subset R$,and let $\overrightarrow{T} = (T_1, T_2, \ldots, T_m), T_j = T_j(X_1, X_2, \ldots, X_n), j = 1, 2, \ldots, m.$ be **any** statistic. Let $U = U(X_1, X_2, \ldots, X_n)$ be an unbiased statistic for θ. Prove or disporve that
   1. $E_\theta(U|\overrightarrow{T})$ is an unbiased statistic for θ. (4 points)
   2. $\sigma^2_\theta[E_\theta(U|\overrightarrow{T})] \le \sigma^2_\theta[\overrightarrow{U}]$. (3 points)

2. Let $X_1, X_2, \ldots, X_n$ be independent random variables with p.d.f. f given by

$$f(x;\theta) = \frac{1}{\theta}e^{-x/\theta}, \quad x > 0, \theta > 0.$$

   1. Derive(base on Neyman-Pearson Lemma) the uniformly most powerful test for testing the hypothesis $H: \theta = \theta_0$ against the altenative $A: \theta < \theta_0$ at level of significance α. (8 points)
   2. Determine the minimum sample size n required to obtain power at least 0.95 against the alternative $\theta = 500$ when $\theta_0 = 1000$ and $\alpha = 0.05$.(7 points)