臺灣大學數學系

八十七學年度第二學期碩博士班資格考試試題

統計與機率

Probability

1. Let $X_1, \cdots, X_n$ be a random sample from logistic distribution with cdf $F(x) = 1/(1+e^{-x})$. Let $V_n = \max(X_1, \cdots, X_n)$.

   (11) Show $V_n \xrightarrow{P} \infty$.

   (12) Show $V_n - \log n$ converge to a limiting distribution.

   (13) Find $\lim_{n\to\infty} P(V_n - \log n \le 0)$.

2. Let $\{N_k\}$ be a sequence of positive, integral-valued random variables such that $k^{-1}N_k \xrightarrow{P} c$ as $k \to \infty$, where $0 < c < \infty$. Let $\{X_n\}$ be a sequence of independent, identically distributed random variables with $EX_n = 0$ and $EX_n^2 = 1$, $n \ge 1$. Find the asymptotic distribution of $\sum_{j=1}^{N_n} X_j/\sqrt{N_n}$ as $n \to \infty$. Justify your answer.

3. Let $\{X_n\}$ be a sequence of random variables satisfying

$$X_1 > X_2 > \cdots > 0 \text{ almost surely.}$$

   Show that $X_n \xrightarrow{a.s.} 0$ if $X_n \xrightarrow{P} 0$

4. Suppose that X and Y are independent random variables with a common distribution function F that is positive and continuous. What is the conditional probability of $[X \le x]$ given the random variable $M = max(X, Y)$?

Statistics

1. Let $X_1, \ldots, X_n$ be a random sample from a population with density $f(x, \theta) = \theta(\theta+1)x^{\theta-1}(1-x)$, $0 < x < 1$, $\theta > 0$.

   1. Show that $T_n = \frac{2\bar{X}}{1-\bar{X}}$ is a method of moment estimate of θ.

   2. Show that

$$\frac{\sqrt{n}(T_n - \theta)}{\theta(\theta+2)^2/2(\theta+3)} \to N(0,1)$$

in law of large number.

2. Let $Y_{ij} = \beta_i + \epsilon_{ij}$, $1 \le j \le n_i$, $i = 1, \cdots, p$, where the ϵ_{ij} are independent $N(0, \sigma_i^2)$ variables, $i = 1, \cdots, p$. Derive the likelihood ratio test of $H_0 : \sigma_1^2 = \cdots = \sigma_p^2$ versus $H_a : \sigma_i^2 \ne \sigma_j^2$ for some i, j is used.