ENTRANCE EXAM FOR PHD PROGRAM

JUNE 3, 2005

All random variables considered here are given on a probability space $(\Omega, \mathcal{F}, P)$.

1. (25 pts) Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed (i.i.d.) random variables taking only positive values. Let $S_n = \sum_{j=1}^{n} X_j$ and $N_t = \sup\{n \,:\, S_n \leq t\}$ for $t > 0$.

   (a) Suppose $EX_1 = \mu \in (0, \infty]$. Prove that $\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mu}$ almost surely (a.s.), where $1/\infty = 0$.

   (b) Suppose $\mu < \infty$ and $\sigma^2 = Var(X_1) < \infty$. Using the fact that the convergence in central limit theorem is uniform, show that $\frac{N_t - t\mu^{-1}}{\sqrt{t\sigma^2\mu^{-3}}}$ converges in distribution as $t \to \infty$ to $N(0, 1)$, the normal distribution with mean 0 and variance 1.

2. (25 pts)

   (a) Suppose that random variables $X_n \to X$ in probability and f is a continuous function. Show that $f(X_n) \to f(X)$ in probability.

   (b) Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with finite mean μ and finite variance. Show that

$$\lim_{n\to\infty} \binom{n}{2}^{-1} \sum_{1\leq i<j\leq n} X_i X_j = \mu^2 \quad \text{in probability.}$$

3. (25 pts) Let $X_n, n \geq 0$, be a martingale and $H_n, n \geq 1$, be a predictable sequence with respect to the filtration $\mathcal{F}_n, n \geq 0$. Show that

$$(H \cdot X)_n = \sum_{j=1}^{n} H_j (X_j - X_{j-1})$$

   is a martingale provided $E[H_n^2(X_n^2 - X_{n-1}^2)] < \infty$ for every n.

4. (25 pts) Let X_n be a martingale with respect to the filtration $\mathcal{F}_n, n \geq 1$, and $S < T$ be two stopping times. Consider the relation

$$E[X_T \,|\, \mathcal{F}_S] = X_S. \tag{1}$$

   (a) Give an example to show that (1) is not true in general.

   (b) Show that (1) is valid when either T is bounded or $|X_n|$ is uniformly bounded, i.e. either $T \leq k \in \mathbb{N}$ a.s. or $|X_n| \leq c \in \mathbb{R}_+$ a.s. for all n.