In the following problems, we use the following convention on the derivatives

$$u_x = \frac{\partial u}{\partial x}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \quad \text{etc.}$$

$D_x u$, $D_x^2 u$ denotes respectively the gradient and the Hessian of u with respect to x.

(A) Consider the Cauchy problem for the Burger's Equation

$$\begin{cases} u_t + (\frac{1}{2}u^2)_x = 0 & \text{for} \quad x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x) & \text{for} \quad x \in \mathbb{R} \end{cases}$$

where $f \in C^3(\mathbb{R})$ and it has non-empty compact support.

(a) Prove that the unique classical C^1 solution $u(x,t)$ exists in $\mathbb{R} \times [0, T^*)$, where $T^* = -(\inf_{x\in\mathbb{R}} f'(x))^{-1}$.

(b) Suppose that $\inf_{x\in\mathbb{R}} f'(x)$ is attained at a single point y_0 with $f'''(y_0) > 0$. Prove that $u(x,t)$ can be uniquely extended to $u(x,T^*)$ which is continuous in $\mathbb{R}$, C^1 in $\mathbb{R} - \{x_0\}$, and $\lim_{x\to x_0} u(x,T^*) = -\infty$, where $x_0 = y_0 + T^* f(y_0)$.

(c) Prove that $(u(x,T^*) - u(x_0,T^*))^3$ ic C^1 in $\mathbb{R}$, and that its derivative at x_0 has the value $-6((T^*)^4 f'''(y_0))^{-1}$.

(B) Find the characteristic curves for the second-order PDE

$$x^2 u_{xx} - 2xy u_{xy} - 3y^2 u_{yy} = 0.$$

And solve this equation with the Cauchy data $u(x,1) = 1$, $u_y(x,1) = x$.

(C) Let $\Omega_\varepsilon = \{(x,y) \in \mathbb{R}^2 | -\frac{\pi}{2} + \varepsilon < y < \frac{\pi}{2} - \varepsilon\}$ where $0 \le \epsilon < \frac{\pi}{2}$. Assume that $u \in C^2(\Omega_\varepsilon) \cap C(\bar{\Omega}_\varepsilon)$ is harmonic in Ω_ε with $u = 0$ on $\partial\Omega_\varepsilon$.

(a) For $\varepsilon = 0$, show by an example that u may not be identically 0.

(b) When $\varepsilon > 0$ and

$$\limsup_{(x,y)\in\Omega_\varepsilon, |x|\to\infty} |u(x,y)| e^{-|x|} = 0,$$

prove that $u = 0$ in Ω_ε.

(c) When $\varepsilon = 0$, must $u = 0$ in Ω_ε if the condition in (b) holds?

(D) Let $u = u(t, x, y) \in C^2(\mathbb{R} \times [0, \infty) \times \times \mathbb{R}^{n-1})$ satisfy the wave equation

$$u_{tt} = c^2 \left(u_{xx} + \sum_{j=1}^{n-1} u_{y_j y_j} \right) \quad \text{in} \quad x > 0,$$

and $u(t, 0, y) = u_x(t, 0, y) = 0$ for $0 \leq t \leq T$ and $y \in \mathbb{R}^{n-1}$, where $T > 0$ is some given time, and $c > 0$ is the wave speed. Apply the HOlmgren uniqueness theorem (or other method)to prove that

$$u = 0 \quad \text{for} \quad (x, t) \quad \text{in} \quad 0 \leq x \leq c \left(\frac{T}{2} - |t - \frac{T}{2}| \right).$$

(E) Let $u = u(x, t) \in C(\mathbb{R}^n \times [0, \infty))$ have classical derivatives u_t, $D_x u$, $D_x^2 u \in C(\mathbb{R}^n \times (0, \infty))$. Assume that for some $0 < p \leq 1$, u satisfies

$$\begin{cases} u_t = \triangle u - |u|^{p-1} u & \text{for } x \in \mathbb{R}^n, t > 0 \\ |u(x, t)| \leq M & \text{for all } (x, t) \end{cases}$$

where M is some positive constant, and use the convention that $|u^{p-1}|u = 0$ if $u = 0$. Assume that $u(x, 0) \geq 0$ for all $x \in \mathbb{R}$.

(a) Prove that $u(x, t) \geq 0$ for all x, t.

(b) For $0 < p < 1$, prove that there exists some finite time $0 < T < \infty$ such that $u(x, t) = 0$ for all $t \geq T$.

(c) For $p = 1$, prove that $u(x, t) > 0$ for all $t > 0$ and $x \in \mathbb{R}$. Find $u(x, t)$ explicitly in terms of $u(x, 0)$.