台灣大學數學系
九十一學年度第二學期博士班資格考試題
偏微分方程

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Choose 4 problems from below.

1. Solve the equations:
(a) $u_x + yu_y - u_z = -u, u(x, y, 0) = x + y.$

(b) $u_x u_y = u, u(x, 0) = x^2.$

2. Let $u \in C^2(R \times [0, \infty))$ solve

$$u_{tt} - u_{xx} = 0, u(x, 0) = f(x), u_t(x, 0) = g(x).$$

Suppose f, g have compact support. Show that

(a) $\int_{-\infty}^{\infty} \{u_t^2 + u_x^2\}\, dx$ is a constant in t,

(b) $\int_{-\infty}^{\infty} \{u_t^2 - u_x^2\}\, dx = 0$ for large t.

3. Let u solve $u_t + 6uu_x + u_{xxx} = 0$ for $x \in R, t > 0$. Suppose u has the form $u(x, t) = v(x - ct)$ for some constant c with $v(s), v'(s), v''(s) \to 0$ as $s \to \infty$ or $s \to -\infty$. Find an explicit formula for u.

4. Let u be a C^2 solution of $\triangle u = 0$ in Ω and $\{x : |x - x_o| \leq \rho\} \subset \Omega$. Show that

(a) $u(x_o) = \frac{1}{\omega_n \rho^{n-1}} \int_{|x-x_o|=\rho} u(x)\, dS_x$, where $\omega_n \rho^{n-1}$ is the surface area of the sphere $|x - x_o| = \rho$.

(b) $|Du(x_o)|^2 \leq \frac{1}{\omega_n \rho^{n-1}} \int_{|x-x_o|=\rho} |Du(x)|^2\, dS_x$.

5. Suppose $f(x)$ is bounded and continuous in R^n which satisfies $\int_{R^n} |f(x)|\, dx < \infty$. Let u be a bounded solution of

$$u_t = \triangle u \quad \text{for } x \in R^n, t > 0; u(x, 0) = f(x).$$

Show that $lim_{t\to\infty} u(x, t) = 0$.

6. Let u be a C^1 solution of $u_t + uu_x = 0$ in each of two regions separated by a smooth curve $x = \xi(t)$. Suppose u is continuous, but u_x has a jump discontinuity on the curve. Prove that

$$\frac{d\xi}{dt} = u(\xi(t), t).$$