臺灣大學數學系

九十一學年度第一學期碩博士班資格考試題

偏微分方程(Differential)

Sept 11, 2002

Total score: 100 points

$1.$

(20 points) Solve the following Cauchy problem for $u(x,y)$:

(a)

(10 points) Equation $(x+2)u_x + 2yu_y = 2u$ with initial condition

$u(-1,y) = \sqrt{y}$.

(b)

(10 points) Equation $x^2u_x - y^2u_y = 0$ with initial condition $u(1,y) = f(y)$.

$2.$

(20 points)(a) (10 points) Show that for $n=3$ the general solution of the wave equation:

$$u_{tt} = c^2 \sum_{i=1}^{n} u_{x_i x_i},$$

with spherical symmetry about the origin has the form:

$$u = \frac{f(r+ct) + g(r-ct)}{r}, \qquad r^2 = \sum_{i=1}^{n} x_i^2,$$

with suitable f and g. Here c is a positive constant. (b) (10 points) Show that the solution of the above problem with initial data of the form:

$$u = 0, \quad u_t = \phi(r)$$

is given by

$$u = \frac{1}{2cr} \int_{r-ct}^{r+ct} \xi \phi(\xi)\, d\xi.$$

Here ϕ is a even function of r.

3.

(20 points) Let u be harmonic in a domain D. Show that u has partial derivatives of all orders in D.

4.

(20 points) Consider the following one-dimensional diffusion equation in the semi-infinite interval $0 \le x < \infty$:

$$u_t - u_{xx} = 0, \qquad x \ge 0, \quad t \ge 0$$

$$u(x,0) = 0, \qquad u(0,t) = at^n \text{ if } \quad t > 0,$$

where a is a positive constant and n is a non-negative constant.

(a)

(10 points) Assume the solution of the problem take the form:

$$u(x,t) = at^n f(\xi), \text{ where } \quad \xi = \frac{x}{2\sqrt{t}}.$$

Show that f satisfies the conditions:

$$f''(\xi) + 2\xi f'(\xi) - 4nf = 0, f(0) = 1, \qquad f(\infty) = 0.$$

(b)

(10 points) Find the solution of the above ordinary differential equation, and hence the solution of the original diffusion problem.

5.

(20 points) Solve the following one-dimensional diffusion equation in the unit interval $0 \le x \le 1$:

$$u_t - u_{xx} = x \sin t, \qquad 0 \le x \le 1, \quad t \ge 0$$

$$u(x,0) = x(1-x), \qquad u(0,t) = u_x(1,t) = 0 \text{ if } \quad t > 0.$$