臺灣大學數學系

九十學年度第一學期碩博士班資格考試試題

微分方程式

1

(20 points) Solve the following problems using characteristic methods:

(a) $uu_{x_1} + u_{x_2} = 1,\ u(x_1, x_1) = \frac{x_1}{2}$

(b) $x_1 u_{x_1} + 2x_2 u_{x_2} = u,\ u(x_1, x_1) = g(x_1)$

where $g \in C^2(R)$.

2

(a) (10 points) Show that the function

$u(x,t) = \frac{1}{\sqrt{t}} f \frac{1}{\sqrt{t}}$

is a solution of the heat equation $u_t = u_{xx}$ in $R \times (0, \infty)$ if and only if f satisfies the following ordinary differential equation

$$f''(\xi) + \xi f'(\xi) + f(\xi) = 0 \qquad \forall \xi \in R \qquad\qquad (*)$$

(b) (10 points) Find all solution of the above ordinary differential equation ($*$). Hence or otherwise find a self-simiilar solution of the heat equation in $R^n \times (0, \infty)$.

3

(20 points) Let $u \in C^2$ for $|x| < a$; $u \in C^0$ for $|x| \le a$; $u \ge 0,\ \triangle u = 0$ for $|x| < a$. Show that for $|\xi| < a$,

$$\frac{a^{n-2}(a-|\xi|)}{(a+|\xi|)^{n-1}} u(0) \le u(\xi) \le \frac{a^{n-2}(a+|\xi|)}{(a-|\xi|)^{n-1}} u(0)$$

4

(20 points) Let $\Omega \supset R^n$ be a bounded domain and let $G(x,y)$ be the Green function for the Laplacian in Ω. That is $\triangle_y G(x,y) = -\delta_x$ in Ω and $G(x,y) = 0$ for any $x \in \Omega$, $y \in \partial\Omega$ where δ_x is the delta mass at x. Prove that

(a) $G(x,y) \ge 0\ \forall x,\ y \in \Omega,\ x \ne y$

(b) $G(x,y) = G(y,x)\ \forall x,\ y \in \Omega,\ x \ne y.$

5

(20 points) Suppose $f \in C(R^n) \cap L^\infty(R^n)$. Show that the function

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{R^n} e^{-|x-y^2/4t|} f(y)dy$$

satisfies the heat equation in $R^n \times (0,\infty)$ and

$$\lim_{t \searrow 0} u(x,t) = f(x) \qquad \forall x \in R^n.$$