臺灣大學數學系

八十九學年度第一學期碩博士班資格考試試題

微分方程式

Choose 4 out of the following 6 problems

1.

Find the solution $u(x,y)$ in each case, defined at least near the initial line $y=0$:

(a)

$$xu_y - yu_x = x, \quad u(x,0) = |x|,$$

(b)

$$u_y - uu_x = 0, \quad u(x,0) = -x.$$

2.

Let Ω be a domain in $\mathbb{R}^3$. Suppose $u(X,t)$ is a smooth solution of the wave equation

$$u_{tt} = c^2 \Delta u, \quad X \in \Omega, \quad t > 0$$

with the boundary condition

$$u(X,t) = 0, \quad X \in \partial\Omega, \quad t > 0.$$

Show that the energy

$$\frac{1}{2}\int_{\Omega} [(u^2)_t + c^2 |\nabla u|^2] \, dX$$

is conserved in time.

3.

Let $u(X) \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be a solution of

$$\Delta u + \sum_{k=1}^{n} a_k(X) u_{X_k} + c(X) u = 0,$$

where $c(X) < 0$ in Ω. Show that $u = 0$ on $\partial\Omega$ implies $u = 0$ in Ω.

4.

Consider the Laplace equation in two space dimensions

$$u_{xx} + u_{yy} = 0$$

in the upper halfplane $y > 0$ with the Dirichlet boundary condition $u(x,0) = f(x)$.

(a)

(b)

Find the solution of this problem.

Show that the solution you find in (a) actually represents a bounded solution of the Dirichlet problem under study, if $f(x)$ is bounded and continuous.

5.

Let $u(x,t)$ be the solution of

$$u_t = u_{xx}, \quad -\infty < x < \infty, \quad u(x,0) = f(x),$$

where f is continuous and $f(x) = 0$ for $|x| \geq 1$. Show that there is a number M, not depending on f, so that

$$\left|\frac{\partial u}{\partial x}(x,t)\right| \leq M \int_{-1}^{1} |f(s)|\, ds$$

for all $t \geq 1$ and all x.

6.

Solve the initial-boundary value problem of the heat equation in two space dimensions,

$$\begin{aligned} u_t &= u_{xx} + u_{yy} && \text{for } 0 < x < 1, 0 < y < 1, t > 0 \\ u(x,0,t) &= u(x,1,t) = 0 && \text{for } 0 \leq x \leq 1, t \geq 0 \\ u(0,y,t) &= u(1,y,t) = 0 && \text{for } 0 \leq y \leq 1, t \geq 0 \\ u(x,y,0) &= f(x,y) && \text{for } 0 \leq x \leq 1, 0 \leq y \leq 1. \end{aligned}$$