臺灣大學數學系

八十七學年度第二學期碩博士班資格考試試題

微分方程式

We assume the given functions are all C^∞ if not stated in other way.

1. (5%) Solve $\begin{cases} u_t + cu_x = 0, \quad u = u(t,x), \text{ is constant}, \\ u(0,x) = f(x). \end{cases}$

2. (10%) Solve $\begin{cases} u_t + uu_x = 0, \quad u = u(t,x), \\ u(0,x) = h(x), \end{cases}$

   show that u_x will become infinite in finite time t.

3. (10%) (i) Solve $\begin{cases} \frac{\partial^2}{\partial\xi\partial\eta} u(\xi,\eta) = F(\xi,\eta), \\ u(\eta,\eta) = 0, \ u_\eta(\eta,\eta) = 0, \ \text{for all} \ \eta \in \mathbb{R}. \end{cases}$

   (ii) Use (i), solve $\begin{cases} u_{tt} - u_{xx} = G(t,x), \ u = u(t,x) \\ u(0,x) = 0 \\ u_t(0,x) = 0, \end{cases}$

   (iii) Solve $\begin{cases} u_{tt} - u_{xx} = G(t,x), \ u = u(t,x) \\ u(0,x) = f(x) \\ u_t(0,x) = g(x). \end{cases}$

   Hint: use linear property.

4. (10%) Find by power series expansion with respect to y the solution of the initial-value problem:

$$\begin{cases} u_{yy} = u_{xx} + u, \ u = u(x,y) \\ u(x,0) = e^x, \quad u_y(0,x) = 0. \end{cases}$$

5. (10%) Let f be a scalar analytic function of $z \in \mathbb{C}$. Put $z = x + iy$ with $x, y \in \mathbb{R}, \ f(z) = u(x,y) + iv(x,y)$. [You need to choose a definition of analytic property.]
   1. Show that u, v satisfy the system of Cauchy-Riemann equations:

      $u_x = v_y, \ u_y = -v_x.$

2. Show that $u(x,y), v(x,y)$ are real analytic in (x,y) variables.

(Hint: Use $|a|+|b| \leq \sqrt{2}|a+ib|$ for $a,b \in \mathbb{R}$)

6. (15%) Let L be a differential operator operator from $C^\infty(\mathbb{R},\mathbb{R})$ to itself, the formal adjoint $\widetilde{L}$ is defined as $\int\int v(Lu)dxdy = \int\int(\widetilde{L}v)udxdy$, for all $u,v \in C^\infty$.

Compute the formal adjoint of the following cases:

1. $L[u] = (\frac{\partial}{\partial y} - a(x,y)\frac{\partial}{\partial x})[u] = u_y + a(x,y)u_x$,
2. $L[u] = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})[u] = u_{xx} + u_{yy}$,
3. $L[u] = a(x,y)u_{xx} + b(x,y)u_{yy}$.

7. (15%)

1. Let $u = u(x,y), \Delta u = u_{xx} + u_{yy},\ v(r,\theta) = u(r\cos\theta, r\sin\theta)$. Find the Laplace operator L in polar coordinates $Lv = \Delta u$.
2. Let $f(\theta)$ be a C^∞-function of period 2π with Fourier Series $f(\theta) = \sum\limits_{n=-\infty}^{\infty} A_n e^{in\theta}$. Prove that $v(r,\theta) = \sum\limits_{n=-\infty}^{\infty} A_n r^{|n|} e^{in\theta}$ represents in polar coordinates r,θ the solution of the Laplace equation $Lv = 0$ in the disk $r < 1$ with boundary values f.
3. Derive Poisson's integral formular for v by substituting for A_n their Fourier expressions in terms of f (i.e. $A_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta$) and interchanging summation and integration.

8. (10%) Solve $\begin{cases} u_{tt} - u_{xx} = 0 \\ u_t(0,x) = g(x), g \in \mathcal{S}(\mathbb{R}) \text{ [Schwartz function class]} \\ u(0,x) = 0 \end{cases}$

by performing Fourier transform with respect to the variable x

$$\hat{u}(t,\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi}u(t,x)dx.$$

9. (15%) (i) Solve $\begin{cases} u_t = u_{xx} \\ u(0,x) = f(x), f \in \mathcal{S}(\mathbb{R}) \quad \text{[Schwartz function class]} \end{cases}$

by performing Fourier transform with respect to the variable $x,\ \hat{u}(t,\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi}u(t,x)dx$, show $u(t,x) = \frac{1}{\sqrt{2\pi}}\int e^{ix\xi - |\xi|^2 t}\hat{f}(\xi)d\xi$.

(ii) Show that $u(t,x)$ may be expressed as

$$u(t,x) = \int K(x,y,t)f(y)dy$$

$$K(x,y,t) = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}$$

(iii) Show that $\lim_{t \to 0} K(x,y,t) = \delta_y$ $[\text{Dirac function}]$ in the sense of distribution.