臺灣大學數學系

八十六學年度第一學期碩博士班資格考試試題

幾何

Statements without proof or counter-example will be considered ``non-mathematical'' statements. You might lose point for making such a statement unless you think that statement is too trivial to explain. Now you have 180 minutes.

1. $\mathbb{RP}^2 = \{(x:y:z)\}$. Is $\mathbb{RP}^2$ an orientable manifold? If not orientable; can you still find a Morse function so that all its critical points have non-degenerate Hessian? (20/100)
2. Cycloid $= \{(x,y) | x = t - \sin t, y = 1 - \cos t, 0 \le t \le 2\pi\}$. Find envelop of normal lines to the cycloid. (20/100)
3. Riemannian metric $ds^2 = dx^2 + dy^2 + \frac{(xdx+ydy)^2}{x^2+y^2} = \frac{2x^2+y^2}{x^2+y^2}dx^2 + \frac{2xydxdy}{x^2+y^2} + \frac{x^2+2y^2}{x^2+y^2}dy^2$, $(x,y) \ne (0,0)$.
   Geoderic $\gamma = (x(t), y(t))$, $\gamma(0) = (1,0)$, $\gamma'(0) = (x'(0), y'(0)) = (0,1)$, $\gamma = ?$ (20/100)
4. For $(x,y) \ne (0,0)$. Surface $\sum = \{z =$ arctan $y/x\}$. Can you extend $\sum$ to the point $(x,y,z) = (0,0,1)$ continuously differentiably? If you can further extend it twice continuously differentiably, find Gauss curvature and mean curvature of $\sum$ at $(x,y,z) = (0,0,1)$. (20/100)
5. $S^2 = 2$-dimensional sphere $= \{x^2+y^2+z^2 = 1\}$, $T^2 = 2$-dim torus $= \{(u,v)\}/\{u \equiv u \pm m, v \equiv v \pm n, m,n \in \mathbb{Z}\}$. Can you find a covering map $\pi : T^2 \longrightarrow S^2$ so that each point $p \in S^2$ has a neighborhood $U \ni p$, $\pi^{-1}(U) =$ disjoint union of U_i, each U_i homeomorphic to U. If yes, $x = x(u,v) = ?$, $y = y(u,v) = ?$, $z = z(u,v) = ?$ (20/100)