

**Entering Examination (Ph.D program) for Discrete Mathematics**  
**Department of Mathematics, National Taiwan University (June 2005)**

1. (20 %) Suppose  $S$  is a set of size  $n$ , where  $n$  is a multiple of 8. Find the number of subsets of  $S$  with size divisible by 4.
  
2. (20 %) A family  $\mathcal{F}$  of sets is called a *Sperner family* if no member of  $\mathcal{F}$  properly contains any other. Let  $\mathcal{F}$  be a family of subsets of an  $n$ -element set  $X$ . Define  $b(\mathcal{F})$  to be the family of all subsets  $Y$  of  $X$  such that (i)  $Y \cap F \neq \emptyset$  for all  $F \in \mathcal{F}$ ; (ii)  $Y$  is minimal subject to (i) (i.e., no proper subset of  $Y$  satisfies (i)).
  - (a) Prove that  $b(\mathcal{F})$  is a Sperner family.
  - (b) Show that, for any  $F \in \mathcal{F}$  and any  $y \in F$ , there exists  $Y \in b(\mathcal{F})$  with  $Y \cap F = \{y\}$ .
  - (c) Deduce that  $b(b(\mathcal{F})) = \mathcal{F}$ .
  - (d) Let  $\mathcal{F}_k$  denote the Sperner family of all  $k$ -element subsets of  $X$ . Prove that  $b(\mathcal{F}_k) = \mathcal{F}_{n+1-k}$  for  $k > 0$ . What is  $b(\mathcal{F}_0)$ ?
  
3. (20 %) Let  $G$  be a connected graph with  $n$  vertices. Define a new graph  $G'$  having one vertex for each spanning tree of  $G$ , with vertices adjacent in  $G'$  if and only if the corresponding trees have exactly  $n(G) - 2$  common edges. Prove that  $G'$  is connected. Determine the diameter of  $G'$ .
  
4. (20 %) An acyclic orientation of a loopless graph is an orientation having no cycle. For each acyclic orientation  $D$  of  $G$ , let  $r(D) = \max_C [a/b]$ , where  $C$  is a cycle in  $G$  and  $a, b$  count the edges of  $C$  that are forward in  $D$  or backward in  $D$ , respectively. Fix a vertex  $x \in V(G)$ , and let  $W$  be a walk in  $G$  beginning at  $x$ . let  $g(W) = a - b \cdot r(D)$ , where  $a$  is the number of steps along  $W$  that are forward edges in  $D$  and  $b$  counts the number that are backward in  $D$ . For each  $y \in V(G)$ , let  $g(y)$  be the maximum of  $g(W)$  such that  $W$  is an  $x, y$ -walk (assume that  $G$  is connected).
  - (a) Prove that  $g(y)$  is finite and thus well-defined, and use  $g(y)$  to obtain a proper  $1 + r(D)$ -coloring of  $G$ . Thus  $G$  is  $1 + r(D)$ -colorable.
  - (b) Prove that  $\chi(G) = \max_D 1 + r(D)$ , where  $D$  runs over all acyclic orientations of  $G$ .
  
5. (20 %) The  $k$ th power of a graph  $G = (V, E)$  is the graph  $G^k = (V, E^k)$ , where  $E^k = \{uv : 1 \leq d_G(u, v) \leq k\}$ . Prove that the cube  $G^3$  of a connected graph  $G$  with at least three vertices is Hamiltonian. (Hint: Reduce to the case of trees, and prove it for trees by proving a stronger result that if  $xy$  is an edge of the tree  $T$ , then  $T^3$  has a Hamiltonian cycle using the edge  $xy$ .)