台灣大學數學系

九十二學年度第二學期博士班資格考試題

實分析

May 9, 2004

There are Problems A to D with a total of 110 points.**Please write down your proof or computatioinal steps clearly on the answer sheets**.

A.

Let $(X, \mathcal{M}, \mu)$ be a measure space where $\mathcal{M}$ is a σ-algebra of subsets of X, and $\mu : \mathcal{M} \to [0, \infty]$ is a measure. Define

$$\mu^*(E) = \inf\{\mu(A) | A \in \mathcal{M} \text{ , and } E \subset A\} \text{ , for any subset } E \subset X$$

(a)

(10 points) Prove that μ^* is an outer measure define in X such that every $E \in \mathcal{M}$ is μ^*-measurable, and $\mu^*(E) = \mu(E)$. Are sets in $\mathcal{M}$ the only μ^*-measurable subset of X?

(b)

(15 points) Let $E_n \subset X (n = 1, 2, 3, \cdots)$ be a sequence of subsets. Probe that $\mu^*(\liminf_{n\to\infty} E_n) \leq \liminf_{n\to\infty} \mu^*(E_n)$. Give example to show that the equality is in general false. Is $\mu^*(\limsup_{n\to\infty} E_n) \geq \limsup_{n\to\infty} \mu^*(E_n)$ also true?

B.

(15 points)Let $f \in L^1(\mathbb{R}^n)$ and $K \subset \mathbb{R}^n$ be a compact subset of $\mathbb{R}^n$. Define the functions

$$\phi(x) = \int_{\mathbb{R}}^{n} |f(x+y) + f(y)| dy, \psi(x) = \int_{k+x} f(y) dy \quad \text{for } x \in \mathbb{R}^n$$

where $K + x = \{y + x | y \in K\}$. Prove that ϕ and ψ are both continuous functions in $\mathbb{R}^n$ .Are both uniformly continuous in $\mathbb{R}^n$?

C.

Let $(X, \mathcal{M}, \mu)$ be a measure space (as in Problem A). Let $f_n (n = 1, 2, 3, \cdots)$ and f be in $L^p(X, \mathcal{M}, \mu)$ with $1 \leq p < \infty$.

(a)

(15 points) If $\lim_{n\to\infty} ||f_n||_{L^p} = ||f||_{L^p}$, prove that $f_n \to f$ in measure iff $\lim_{n\to\infty} ||f_n - f||_{L^p} = 0$

(b)

(15 points) If $||f_n||_{L^p}$ is bounded in L^p and $f_n \to f$ in measure where $1 < p < \infty$, prove that

$$\lim_{n\to\infty} \int_X f_n(x)g(x)d\mu(x) = \int_X f(x)g(x)d\mu(x) \quad \text{for any } g \in L^q(X, \mathcal{M}, \mu)$$

where q is given by $\frac{1}{p} + \frac{1}{q} = 1$. Show that by an example that the conclusion is false in case $p = 1$.

D.

Determine which of the following statements is true or false. Prove your answer. Each has 10 points.

(a)

Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation. Then f is absolutely continuous in $[a, b]$ iff f is absolutely continuous in $[a + \varepsilon, b]$ for all sufficiently small $\varepsilon > 0$.

(b)

Let $E \subset \mathbb{R}^2$ be a Borel subset, then the set $F = \{\sin(x^2 + y^2) | (x, y) \in E\}$ is also Borel in $\mathbb{R}$. However, if E is Lebesgue measurable, the F may not be Lebesgue measurable.

(c)

There exists a collection $\mathcal{F}$ of closed rectangles in $\mathbb{R}^n$ such that $\cup_{E\in\mathcal{F}} E$ is not Lebesgue measurable.

(d)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be Borel measurable. Then, for each $y \in \mathbb{R}$ fixed, the function $f(x, y)$ is also Borel measurable in $x \in \mathbb{R}$.