臺灣大學數學系

九十一學年度第一學期碩博士班資格考試題

分析(Analysis)

Sept 11, 2002

Points distributions: A(40), B(30), C(15), D(15)

A.

Let $[a, b]$ be a closed interval of $\mathbb{R}$. Define the spaces

$B[a, b] = \{f \mid f : [a, b] \to \mathbb{R} \text{ is bounded } \}$

$BV[a, b] = \{f \mid f : [a, b] \to \mathbb{R} \text{ is a function of bounded variation}\}$

For $f \in B[a, b]$, denote $\| f \|_\infty = \sup_{x\in[a,b]} \mid f(x) \mid$ to be the sup-norm of f.

(a)

Prove that $B[a, b]$ with the sup-norm is a Banach space, and $BV[a, b] \subset B[a, b]$. Must $BV[a, b]$ be a closed subspace of $B[a, b]$?

(b)

Prove that every $f \in BV[a, b]$ is Lebesgue measurable, and $BV[a, b] \subset L^1([a, b])$. Must every $f \in BV[a, b]$ be Riemann integrable? If yes, is the Riemann integral of f equal to the Legesgue integral of f?

(c)

Let $f_n \in BV[a, b]$ $(n = 1, 2, 3, \cdots)$ be a sequence, and $\lim_{n\to\infty} f_n = f \in BV[a, b]$ uniformly. Let Γ_n and Γ be the arclength of the graphs of f_n and f respectively. Must $\lim_{n\to\infty} \Gamma_n = \Gamma$ be true?

(d)

Must $B[a, b] \subset \cap_{p\geq 1} L^p([a, b])$? If no, what is $\cap_{p\geq 1} L^p([a, b])$?

B.

Determine which of the following statements are true or false. Prove your assertion.

(a)

Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$. Then, even when $A \times B$ is Legesgue measurable, A and B are not necessarily Legesgue measurable. But if $A \times B$ is Borel measurable, then A and B must be Borel measurable.

(b)

Let $E_m \subset \mathbb{R}^n$ be a sequence of Lebesgue measurable sets, then

$$\lim_{n\to\infty} \mid A - \cup_{j=1}^{m} E_j \mid_e = \mid A - \cup_{j=1}^{\infty} E_j \mid_e$$

holds for any $A \subset \mathbb{R}^n$ with $\mid A \mid_e < \infty$. The notation $\mid A \mid_e$ denotes the Lebesgue outer measure of A.

(c)

Let $E \subset \mathbb{R}^n$ be Lebesgue measurable and $f_m \in L^1(E)$. Assume that $f_1 \geq f_2 \geq \cdots \geq f_m \cdots \geq 0$ a.e. in E, and $f_m \to 0$ a.e. in E, then $\sum_{m=1}^{\infty}(-1)^{m-1} f_m \in L^1(E)$, and

$$\int_E \left(\sum_{m=1}^{\infty}(-1)^{m-1} f_m(x)\right) dx = \sum_{m=1}^{\infty}(-1)^{m-1} \int_E f_m(x) dx.$$

C.

Let $E \subset \mathbb{R}^n$ be Lebesgue measurable and f_m be a sequence of measurable functions in E ocnverging a.e. to f. Assume that there exists another measurable sequence g_m in E such that $0 \leq \mid f_m(x) \mid \leq g_m(x)$ a.e. in E for all m, and g_m converges in $L^1(E)$. Prove that $\lim_{m\to\infty} \int_E f_m(x) dx = \int_E f(x) dx$.

D.

Let $f \in L^p(E) \cap L^q(E)$ with $1 \leq p \leq q \leq \infty$ where E is a Lebesgue measurable set.

(a)

Prove that $\| f \|_r \leq \| f \|_p^{\alpha} \| f \|_q^{1-\alpha}$ for all $p \leq r \leq q$, where $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$.

(b)

prove that, for all $\epsilon > 0$, $\| f \|_r \leq \epsilon \| f \|_p + \epsilon^{-\mu} \| f \|_q$, where $\mu = \frac{\alpha}{1-\alpha}$.