臺灣大學數學系

八十八學年度第一學期碩博士班資格考試試題

分析

*任選五題,但3及10必選

1.

Suppose that (Ω, σ, μ) is a measure space and $A_n \in \Sigma$, $n = 1, 2, 3, \cdots$. Let E={ $x \in \Omega : x \in A_n$ for infinitely many n}.

(i) Show that E= $\bigcap_{k=1}^{\infty} \bigcup_{n \geq k} A_n$.

(ii) Show that μ(E)=0 if $\sum_n \mu(A_n) < +\infty$.

2.

Evaluate $\int_{-\infty}^{\infty} e^{-x^2} dx$ by showing that

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \int_{\mathbb{R}^2} e^{-(x^2+y^2)} dx dy = \int_0^{\infty} 2\pi t e^{-t^2} dt$$

3.

Let $\Omega = (0, \infty) \times (0, \infty)$ and $f(x, y) = e^{-xy}, g(x, y) = e^{-xy} \sin x$ for $(x, y) \in \Omega$.

(i) Show that both f and g are not Lebesgue integrable on Ω.

(ii) From $\int_0^N [\int_0^M e^{-xy} \sin x dy] dx = \int_0^M [\int_0^N e^{-xy} \sin x dx] dy$ for

$N > 0, M > 0$. Show that

$$\int_0^N \frac{\sin x}{x} dx = \int_0^{\infty} \left[\int_0^N e^{-xy} \sin x dx\right] dy$$

(iii) Following (ii) show that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \int_0^{\infty} \left[\int_0^{\infty} e^{-xy} \sin x dx\right] dy$$

(iv) Evaluate $\int_0^{\infty} e^{-xy} dx$ and show that $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$.

4.

Suppose that f is a Lebesgue integrable function on $\mathbb{R}$ and $\varphi \in C^1(\mathbb{R})$. Assume that both φ and φ' are bounded functions. Define

$$F(x) = \int_{-\infty}^{\infty} f(y)\varphi(x-y)dy$$

Show that $F \in C^1(\mathbb{R})$ and

$$F'(x) = \int_{-\infty}^{\infty} f(y)\varphi'(x-y)dy$$

5.

State Hölder's inequality and use it to prove Minkowski's inequality.

6.

Let $K \subset C[0,1]$ be a family of absolutely continuous functions. Suppose that

1) $\sup_{f \in K} |f(0)| < \infty$,

2) $\int_0^1 |f'(x)|^2 dx \leq 1$ for all $f \in K$.

Show that K is precompact in $C[0,1]$.

7.

Let X be a Hilbert space and let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal system in X. Consider $x = (x_i)_{j=1}^{\infty} \in l^2$ and for each n let $V_n = \sum_{j=1}^{n} x_j e_j$. Show that $\lim n\infty V_n$ exists in X and $|| \lim n\infty V_n ||^2$ = $\sum_{j=1}^{\infty} |x_j|^2$.

8.

Prove that ``Monotone convergence Theorem'' for integrals is equivalent to ``Fatou's Lemma'' .

9.

Let f be a function of bounded variation on [0,1].

(i) Show that f is measurable.

(ii) Is it true that the total variation of f is given by $\int_0^1 |f'(x)|dx$?

10.

Let $\{f_n(z)\}_{n=1}^{\infty}$ be a sequence of analytic functions for $|z| < 1$. Suppose that $f_n(z)$ converges uniformly to $f(z)$ on each compact subset of $\{z \in \mathbb{C} : |z| < 1\}$. Show that if each f_n is 1-1 , then so is f unless f is a constant function.