臺灣大學數學系

八十七學年度第二學期碩博士班資格考試試題

分析

A. Choose 4 from the following 5 problems

1. Let f_n and f be measurable functions on $[0,1]$. Assume $\lim_{n\to\infty}\int_0^1 |f_n(x)-f(x)|^p\,dx = 0$ for some $p>0$. Show that f_n converges in measure to f on $[0,1]$.
2. Let Z be a measurable set in $[0,\infty)$ and $E=\{\sqrt{x}(1-x) : x\in Z\}$.

   (a) Show that E is measurable.
   (b) Show that E has measure zero if Z has measure zero.
3. Let $\{r_1, r_2, \ldots, r_k, \ldots\}$ be all of the rational numbers in $[0,1]$. Let $f_n(x)=\sum_{k=1}^{n}\frac{1}{k^3|x-r_k|^{1-\frac{1}{k}}}$.
   (a) Show that $\lim_{n\to\infty} f_n(x)$ exists and $<\infty$ a.e. on $[0,1]$. (b) Show that for a.e. $y\in[0,1]$, we have $\lim_{n\to\infty}[f_n(x)f_n(x+y)]$ exists and $<\infty$ for a.e. $x\in[0,1]$.
4. Let f be a measurable function on $[a,b]$. Determine which of the following conditions implies that
   $$f(x)=\int_a^x f'(t)\,dt + f(a) \text{ for all } x\in[a,b].$$
   (1) $|f(x)-f(y)|\le L|x-y|$ for all $x,y\in[a,b]$. (2) $|f(x)-f(y)|\le L\sqrt{|x-y|}$ for all $x,y\in[a,b]$.
5. Show that every nonempty, closed, convex set in a Hilbert space contains a unique element of smallest norm.

B. Choose 1 from the following 2 problems

1. Suppose f is an entire function and n is a nonnegative integer. Show that if
   $$f(z)\le a+b|z|^n$$
   for some positive constants a and b, then f is a polynomial of degree at most n.
2. Evaluate the intrgral

$$\int_0^{\infty} \frac{x sin x \, dx}{9 + x^2}.$$