臺灣大學數學系

八十六學年度第二學期碩博士班資格考試試題

分析

There are problems A to F. You have to do Problems B, E, F, and 2 Problems out of A, C, and D.

A.

Let $f:[0,1]\times[0,1]\to\mathbb{R}$, $f=f(x,y)$. If f is measurable with respect to x for each fixed y, and f is continuous in y for almost everywhere fixed x, prove that f is measurable. Is the conclusion true if we only assume that f is measurable in x and y separartely?

B.

Let $E\subseteq\mathbb{R}^n$ be measurable, and $f_m (m=1,2,\cdots)$ be a sequence of functions in $L^p(E)$ with $1\le p\le\infty$. For an $f\in L^p(E)$, we have the following 4 possible ways of convergence:

(a)

$f_m\to f$ a.e. ;

(b)

$f_m\to f$ in measure ;

(c)

$f_m\to f$ in L^p, i.e. , $\lim\limits_{m\to\infty}\int_E|f_m(x)-f(x)|^p dx=0$;

(d)

$f_m\to f$ weakly, i.e. for all

$g\in L^q(E), \lim\limits_{m\to\infty}\int_E f_m(x)g(x)dx=\int_E f(x)g(x)dx$,

where $\frac{1}{p}+\frac{1}{q}=1$.

(1)

Prove that $(a)\Rightarrow(b)$, $(c)\Rightarrow(b)$, and $(c)\Rightarrow(d)$. In each case, show by example that the converse implication is false.

(2)

If $\lim_{m\to\infty}\int_E|f_m(x)|^p dx=\int_E|f(x)|^p dx$. Prove that $(b)\Leftrightarrow(c)$ and $(a)\Leftrightarrow(d)$.

C.

Determine which of the following conditions implies that

$$f(x) = \int_a^x f'(t)dt + f(a), \quad \text{for all} \quad x \in [a, b]$$

where $f : [a, b] \to \mathbb{R}$ is a function of bounded variation.

(1)

$|f(x) - f(y)| \le L\sqrt{|x-y|}$ for all $x, y \in [a, b]$.

(2)

f is differentiable at every point of (a, b), and $|f'(x)| \le L$ for all $a < x < b$.

(3)

f is differentiable at every point of $[a, b)$, f' is bounded in $[a, b-\epsilon]$ for all small $\epsilon > 0$, and the improper Riemann integral of f' on $[a, b]$ exists.

Here L is a positive constant.

D.

Let $f \in L^p(\mathbb{R}^n)$ with $1 \le p \le \infty$. Define

$$u(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{y^2 + (x-t)^2} f(t)dt, \quad y > 0.$$

Prove that $u(x, y)$ is C^∞ in the upper half plane $y > 0$, $u_{xx} + u_{yy} = 0$ in $y > 0$, and $\lim_{y \to 0+} u(x, y) = f(x)$ for a.e. $x \in \mathbb{R}^n$.

E.

Let $E \subset \mathbb{R}$ be a compact set with $|E| > 0$. Define

$$f(z) = \int_E \frac{1}{t-z} dt, \quad z \in \Omega$$

where Ω is the complement of E in the complex plane.

(1)

Prove that f is analytic in Ω.

(2)

Is $z = \infty$ a regular point, or a pole, or an essential singularity of f?

(3)

Compute $\int_\Gamma f(z)dz$, where Γ is a positvely oriented simple closed curve in the plane which contains E in its interior.

F.

Determine the range of $t > 0$ such that the improper integral $\int_0^\infty \frac{\log(1+x^2)}{x^t} dx$ exists. If it exists, find its value.