臺灣大學數學系

八十六學年度第一學期碩博士班資格考試試題

分析

There are problems A to E. You have to do problems A, B, C, and one of D, E. For problem A, work out 2 out of the 3 subproblems. For problem B, work out 1 of the 2 subproblems. For problem C, work out 2 of the 4 problems.

A.

Define the following terminologies:

Lebesque outer measure in $\mathbb{R}^n$, Lebesque measure in $\mathbb{R}^n$, Lebesque measurable function, Lebesque integrable function.

Then determine which of the following statements is true. Prove your answer.

(a)

Let E, F be two subsets of $\mathbb{R}^n$. F is Lebesque measurable, and the distance $d(E,F)=0$. Then $|E \cup F|_e = |E|_e + |F|_e$.

(b)

A subset $E \subseteq \mathbb{R}^n$ is measurable iff $|G| = |G \cap E|_e + |G - E|_e$ for all open subsets $G \subseteq \mathbb{R}^n$.

(c)

Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be Lebesque measurable, and $\phi : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Then $\phi(f,g)$ must be measurable on $\mathbb{R}^n$.

B.

State the Fatou Lemma, the monotone convergence theorem, and the Lebesque dominated convergence theorem. Be sure to include all reasonable hypothesis to ensure the truth of the theorems. Then work out the following problems.

(a)

Determine whether the following limits exists. If yes, evaluate it.

$$\lim_{n\to\infty} \int_0^1 (1 - e^{-x^2/n}) x^{-1/2} dx; \quad \lim_{n\to\infty} \int_0^1 f(x)\phi(nx) dx$$

where $\phi(x)$ is the characteristic function of $\bigcup_{k=0}^{\infty} [2k,\ 2k+1]$, and $f \in L^1([0,1])$.

(b)

Suppose f is integrable on $[0,\ \infty)$. Define

$$g(x) = \int_0^{\infty} \frac{f(t)}{x+t} dt, \quad x > 0.$$

Is g continuous? Does g have a limit as $x \to \infty$? Is g differentiable?

C.

Let $f : [a,\ b] \to \mathbb{R}$ be differentiable a.e. on $[a, b]$. Define

$$E = \{x \in [a, b] \mid f'(x) \text{ doesn't exist } \}.$$

(a)

Show that f' must be Lebesque measurable. Must f' be Lebesque integrable when $E = \emptyset$?

(b)

If f is continuous, $f'(x) = 0$ for $x \notin E$, and E is an isolated subset of (a, b), must f be a constant? How about the conclusion if E is assumed to be a closed set in $[a, b]$?

(c)

Show that, if $E = \emptyset$, and f' is bounded, then f is absolutely continuous.

(d)

Assume that f is absolutely continuous, and f' lies in $L^p([a, b])$ for some $1 \le p < \infty$. Prove that there exists a sequence of continuously differentiable functions g_n on $\mathbb{R}$ with compact support such that

$$\lim_{n\to\infty} \int_a^b |f(x) - g_n(x)|^p dx = 0, \quad \lim_{n\to\infty} \int_a^b |f'(x) - g_n'(x)|^p dx = 0.$$

D.

Let D be the unit disc consisting of all complex numbers z with $|z| < 1$. State and prove the Schwarz Lemma for analytic functions defined on D. Then compute

$$\sup\{|f'(\alpha)| \mid f : D \to D \text{ is analytic}\}$$

where $\alpha \in D$ is fixed.

E.

Find the value of

$$\int_0^\infty \frac{\log x}{\sqrt{x}(x^2 + a^2)^2} dx$$

where a is a positive constant.