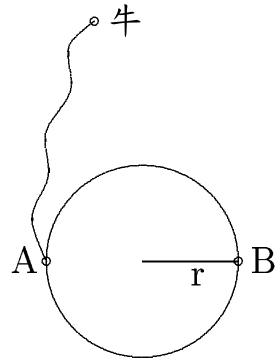
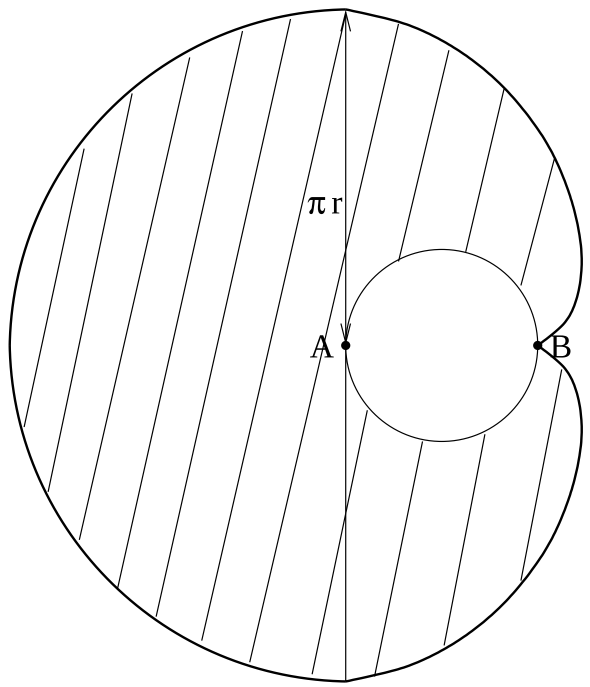
牛吃草問題

有一頭牛, 被栓在一個半徑為r的木樁上(如圖一所示) 繩子的一端被固定在A點, 而牛能夠走到木樁的對面B. 木樁的外部都是草地, 請問牛有辦法吃到多少草?

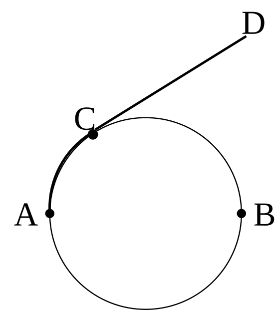


圖一

經由觀察我們發現牛能吃到草的範圍如右圖的斜線部份(見圖二). 由題意知繩長為πr, 而在A點左邊的區域會是一個半圓. 至於剩下的區域怎麼求得呢? 當繩子被木樁"拌住"的時候(見圖三). 牛所達到的最遠處為D, 其中弧長$\widehat{AC}$ 加直線長$\overline{CD}$為πr(繩子的長度), 而曲線即所有這種點所形成的軌跡.

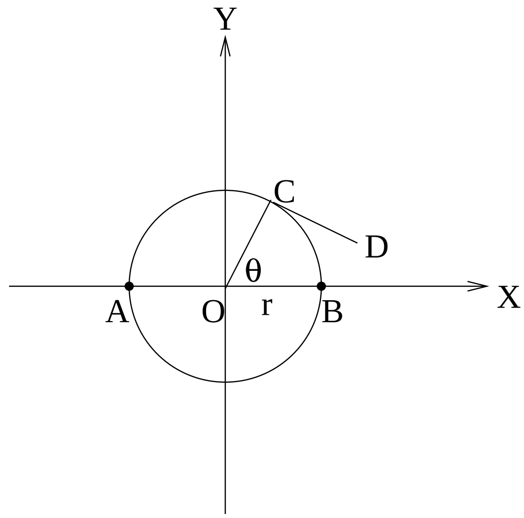


圖二

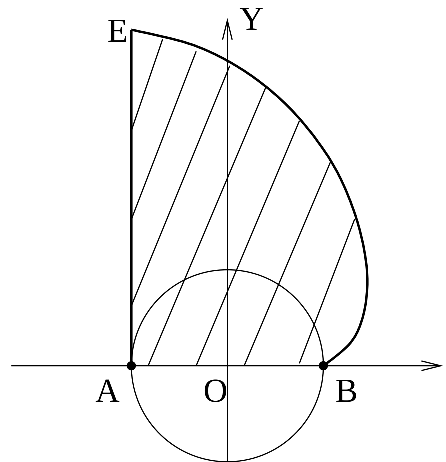


圖三

我們可以利用解析幾何將軌跡描述出來: 取木樁的中心為原點O, 令$\overrightarrow{OB}$與$\overrightarrow{OC}$ 的夾角為θ(如圖四), 於是C點坐標為$(r\cos\theta, r\sin\theta)$,而 $\overline{CD}$會是圓在C點上的切線段, 所以$\overrightarrow{CD} = k(\sin\theta, -\cos\theta)$, k待定, 而$\overline{CD}$長度要等於弧長 $\overset{\frown}{BC}$, 於是$|\overrightarrow{CD}| = r\theta$, 解得 $k = r\theta$, 所以D點坐標即確定: $(r\cos\theta, r\sin\theta) + r\theta(\sin\theta, -\cos\theta) = (r(\cos\theta + \theta\sin\theta), r(\sin\theta - \theta\cos\theta))$



圖四



圖五

我們可先計算圖五的斜線面積, 它會是以下所表示的積分值:

$$
\begin{aligned}
\text{Area} &= \int y(\theta)dx(\theta) = \int y(\theta)x'(\theta)d\theta \\
&= -\int_0^{\pi} r(\sin\theta - \theta\cos\theta)\cdot r\theta\cos\theta d\theta \\
&= -r^2\left[\int_0^{\pi}(\theta)(\sin\theta\cdot\cos\theta)d\theta - \int_0^{\pi}\theta^2\cos^2\theta d\theta\right] \\
&= -r^2[(1)+(2)]
\end{aligned}
$$

$$
\begin{aligned}
(1) &= \tfrac{1}{8}\int_0^{\pi}(2\theta)(\sin 2\theta)d(2\theta) = \tfrac{1}{8}\int_0^{2\pi} x\cdot\sin x dx = -\tfrac{1}{8}\int_0^{2\pi} xd(\cos x) \\
&= -\tfrac{1}{8}[x\cdot\cos x|_0^{2\pi} - \int_0^{2\pi}\cos x dx] = -\tfrac{1}{8}(2\pi) = -\tfrac{\pi}{4}
\end{aligned}
$$

(其中$\cos x$為周期函數, 故$\int_0^{2\pi}\cos x dx = 0$)

$$
\begin{aligned}
(2) &= -\int_0^{\pi}\theta^2\left(\tfrac{1+\cos 2\theta}{2}\right)d\theta = -\int_0^{\pi}\tfrac{1}{2}\theta^2 d\theta - \tfrac{1}{2}\theta^2\cos 2\theta d\theta \\
&= -\tfrac{1}{6}\theta^3|_0^{\pi} - \tfrac{1}{16}\int_0^{\pi}(2\theta)^2\cos 2\theta d(2\theta) \\
&= -\tfrac{1}{6}\pi^3 - \tfrac{1}{16}\int_0^{2\pi} x^2\cos x dx \\
&= -\tfrac{1}{6}\pi^3 - \tfrac{1}{16}[x^2\sin x|_0^{2\pi} - \int_0^{2\pi}\sin x\cdot 2x dx] \\
&= -\tfrac{1}{6}\pi^3 - \tfrac{1}{8}[x\cdot\cos x|_0^{2\pi} - \int_0^{2\pi}\cos x dx] = -\tfrac{1}{6}\pi^3 - \tfrac{\pi}{4}
\end{aligned}
$$

$\therefore \mathrm{Area} = \frac{1}{6}\pi^3 r^2 + \frac{1}{2}\pi r^2$,
至此可得吃草的範圍=上下兩塊Area, 加上左半圓扣掉木樁面積 $= 2\cdot(\frac{1}{6}\pi^3 r^2 + \frac{1}{2}\pi r^2) + \frac{(\pi r)^2\pi}{2} - \pi r^2 = \frac{5}{6}\pi^3 r^2$(平方單位)

補充: 圖五中弧$\overset{\frown}{BE}$稱為<u>圓的漸伸線</u>(involute)