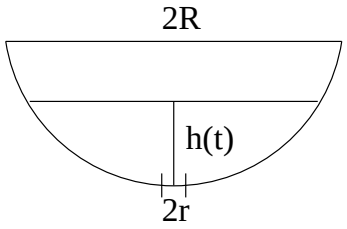
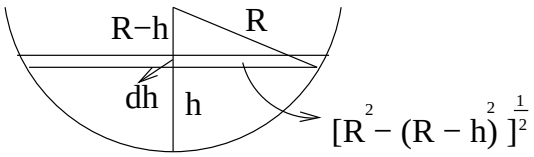
托里切利定律

根據托里切利定律, 一個水分子從桶漏出的速率等於一個水分子從水面自由落體到桶底的速率, 現在一個半圓球狀的碗, 裝滿水, 半徑為R, 碗底有一小圓孔, 半徑是r, 請問, 要多少時間水才會從碗裡漏光?



如上圖, 水面高$h(t)$時, 水從底部要漏出的速率為$\sqrt{2gh}$, ($\frac{1}{2}mv^2 = mgh$, $v = \sqrt{2gh}$), 設水的體積是$\bar{V}$, $\frac{d\bar{V}}{dt} = -\pi r^2\sqrt{2gh}$, $\frac{d\bar{V}}{dh} \cdot \frac{dh}{dt} = -\pi r^2\sqrt{2gh}$(Chain Rule)



$\therefore d\bar{V} = \pi(R^2 - (R-h)^2)dh$
$\Rightarrow \frac{d\bar{V}}{dh} = \pi(R^2 - (R-h)^2) = \pi(2Rh - h^2)$
$\Rightarrow \frac{d\bar{V}}{dh} \cdot \frac{dh}{dt} = -\pi r^2\sqrt{2gh}$
$\Rightarrow \cancel{\pi}(2Rh - h^2)\frac{dh}{dt} = -\cancel{\pi} r^2\sqrt{2gh}$
$\Rightarrow \frac{dh}{dt} = -\frac{r^2\sqrt{2gh}}{2Rh - h^2}$
$\Rightarrow (2Rh^{\frac{1}{2}} - h^{\frac{3}{2}})dh = \left(\frac{2Rh - h^2}{\sqrt{h}}\right)dh = -r^2\sqrt{2g}dt$
$\Rightarrow 2R \cdot \frac{2}{3}h^{\frac{3}{2}} - \frac{2}{5}h^{\frac{5}{2}} = -r^2\sqrt{2g}t + C$
當$t = 0$, $h = R$
$\Rightarrow \frac{4}{3}R^{\frac{5}{2}} - \frac{2}{5}R^{\frac{5}{2}} = C$
$\Rightarrow C = \frac{14}{15}R^{\frac{5}{2}}$
$\therefore$當$h = 0$時, 碗中的水便可漏光
$\therefore t = \frac{C}{r^2\sqrt{2g}} = \frac{14R^{\frac{5}{2}}}{15r^2\sqrt{2g}}$