2nd homework
Due date: 10/02

Let $F$ be either $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$.

**Exercise 1.** Show that $(1, 1, 0, 1), (1, 0, 1, 1), (0, 0, 1, 1)$ and $(0, 1, 0, 2)$ form a basis of $F^4$.

**Exercise 2.** Let $S = \{u_1, u_2, u_3\} \subset F^4$ be a subset of vectors in $F^4$ given by
\[
u_1 = (1, 10, -6, -2), \quad u_2 = (-2, 0, 6, 1), \quad u_3 = (3, -1, 2, 4)\]
Let $W := \text{span}_F S$. Find $\dim_F W$.

**Exercise 3.** Let $V$ be a vector space over $F$ and let $\{v_1, v_2, v_3\}$ be a basis of $V$. Show that
\[
\{v_1 + v_2 + v_3, v_1 + 2v_2 + 4v_3, v_1 + 3v_2 + 9v_3\}
\]
is also a basis of $V$.

**Exercise 4 (Extension lemma).** Let $V$ be a finite dimensional vector space over $F$ with $\dim_F V = n$. Let $v_1, v_2, \ldots, v_k \in V$ be linearly independent vectors. Show that there exists $(n - k)$ vectors $v_{k+1}, v_{k+2}, \ldots, v_n$ such that $\{v_1, v_2, \ldots, v_n\}$ is a basis of $V$.

**Exercise 5.** Let $V$ be a finite dimensional vector space over $F$. Let $W_1$ and $W_2$ be subspaces of $V$. Define
\[
W_1 + W_2 := \{x + y \in V \mid x \in W_1, y \in W_2\}.
\]
(1) Show that $W_1 + W_2$ is a subspace of $V$.
(2) Use the previous exercise *Extension lemma* to show
\[
\dim_F (W_1 + W_2) + \dim_F (W_1 \cap W_2) = \dim_F W_1 + \dim_F W_2.
\]