5th homework  
Due date: 4/8

Recall that a matrix $A \in M_n(R)$ is orthogonal if $AA^t = I_n$, $A \in M_n(C)$ is unitary if $AA^* = I_n$ and $A \in M_n(C)$ is normal if $AA^* = A^*A$. Let $i = \sqrt{-1} \in C$.

Exercise 1. Let $A, B \in M_n(R)$. Show that $A + iB$ is unitary if and only if $\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in M_{2n}(R)$ is orthogonal.

Exercise 2. Let 

$$A = \begin{pmatrix} 1 & i & 1 \\ -i & 1 & i \\ 1 & -i & 1 \end{pmatrix}.$$  

Find $P \in M_3(C)$ with such that $P$ is unitary and $P^*AP$ is diagonal.

Exercise 3. Show that $A \in M_n(C)$ is normal if and only if there exists $P \in C[X]$ such that $A^* = P(A)$.

Exercise 4. Let $V$ be a finite dimensional inner product space over $C$. Let $T : V \rightarrow V$ be a self-adjoint operator. Show that

1. $1 + iT$ are invertible, and then
2. $S := (1 - iT)(1 + iT)^{-1}$ is a unitary operator.

Exercise 5. Let $A = (a_{i,j}) \in M_n(C)$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n \in C$ be the eigenvalues of $A$ (counted with multiplicity). Show that $A$ is normal if and only if 

$$\sum_{1 \leq i,j \leq n} |a_{i,j}|^2 = \sum_{k=1}^{n} |\lambda_k|^2.$$