2nd homework
Due date: 3/17

There are six problems.

Exercise 1. Find a set of three polynomials $p_1(x) = a$, $p_2(x) = b + cx$ and $p_3(x) = d + ex + fx^2$ with $a, b, c, d, e, f \in \mathbb{R}$ such that \{p_1(x), p_2(x), p_3(x)\} is an orthonormal set with respect to the inner product $\langle f, g \rangle = \int_0^2 f(x)g(x)dx$.

Exercise 2. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{C}$. For each $x, y \in V$ show that
\[
\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2;
\]
\[
4\langle x, y \rangle = \sum_{k=0}^{3} i^k \|x + i^k y\|^2.
\]
(recall that $\|x\| := \sqrt{\langle x, x \rangle}$).

Exercise 3. Let $V = \mathbb{R}[x]$ be the space of polynomials with coefficients in $\mathbb{R}$. Let $b > a$ be real numbers. Define the inner product on $V$ by
\[
\langle f, g \rangle = \int_a^b f(x)g(x)dx.
\]
For each positive integer $n$, define
\[
q_{2n}(x) = (x - a)^n(x - b)^n,
\]
\[
p_n(x) = \frac{d^n}{dx^n}(q_{2n}(x)).
\]
(1) Show that
\[
\frac{d^{i-1}q_{2n}}{dx^{i-1}}(a) = \frac{d^{i-1}q_{2n}}{dx^{i-1}}(b) = 0
\]
for all $i = 1, 2, \ldots, n$.
(2) Show that $p_n$ has degree $n$.
(3) Show that $p_1, p_2, \ldots, p_n$ are orthogonal (or perpendicular) to each other.

Exercise 4. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^3$ given by $\langle (a_1, a_2, a_3), (b_1, b_2, b_3) \rangle = a_1b_1 + a_2b_2 + a_3b_3$. Let $v_1 = (1, 0, 1)$, $v_2 = (1, 0, -1)$ and $v_3 = (0, 3, 4)$. Apply the Gram-Schmidt process to \{v_1, v_2, v_3\} to obtain an orthonormal set $\{w_1, w_2, w_3\}$. 
Exercise 5. Let
\[ \Omega := \begin{pmatrix} 6 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \in M_3(\mathbb{R}). \]

Find \( A \in M_3(\mathbb{R}) \) such that \( AA^* = \Omega \) (so \( \Omega \) is positive definite).
(Hint: Use Gram-Schmidt process).

Exercise 6. Let \((V, \langle \cdot, \cdot \rangle)\) be an inner product space. If \( T : V \to V \) is a projection (i.e. \( T^2 = T \)) such that \( \|T(x)\| \leq \|x\| \) for all \( x \in V \), show that \( T \) is an orthogonal projection.
(Hint: You may need to show \( \langle x, y \rangle = 0 \) for all \( x \in \text{Ker} T \) and \( y \in \text{Im} T \).)