1st homework
Due date: 9/25

Let $F$ be either $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$.

Exercise 1. Let $V = \mathbb{R}^2$. Define the addition by

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2y_2),$$

and define the scalar product by

$$\alpha \otimes (x_1, x_2) := (\alpha x_1, x_2).$$

Verify if $V$ is a vector space over $\mathbb{R}$ with $\oplus$ and $\otimes$.

Exercise 2. Let $V = \mathbb{R}^2$. Then $V$ is equipped with the usual addition $+$. Define a scalar product by

$$\alpha \otimes (x_1, x_2) := (x_1, 0), \quad \alpha, x_1, x_2 \in \mathbb{R}.$$

Verify if $V$ is a vector space over $\mathbb{R}$ with the usual $+$ and $\otimes$.

Exercise 3. Let $v_1 = (1, 2, 3)$ and $v_2 = (1, -1, 1)$. Determine if $(5, 1, 9)$ is a linear combination of $v_1$ and $v_2$.

Exercise 4. Let

$$S = \{x^3 + x^2 + x + 1, x^2 + x + 1, x + 1\} \subset F[x]$$

Determine if $-x^3 + 2x^2 + 3x + 3$ and $2x^3 - x^2 + x + 3$ are linear combinations of vectors in $S$. Justify your answer.

Exercise 5. Let $V$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Then $V$ is a vector space over $\mathbb{R}$ endowed with the usual addition and scalar product between real-valued functions. Let

$$S = \{1, x, x^2, \ldots, x^n, \ldots\}$$

be the subset of monomials in $V$. Show that the function $f(x) = \sin x$ is NOT a linear combination of vectors in $S$ over $\mathbb{R}$.

Exercise 6. Let $V$ be a vector space over $F$ and let $W_1, W_2$ and $W_3$ be subspaces of $V$. Suppose that

$$W_3 \subset W_1 \cup W_2.$$ 

Show that either $W_3 \subset W_1$ or $W_3 \subset W_2$.

Exercise 7. Let $V$ be a vector space over $F$. Show that if a subset $\{v_1, v_2, \ldots, v_n\}$ of $V$ is linearly independent over $F$, then so is the set $\{v_1 - 2v_2, v_2 - 2v_3, \ldots, v_{n-1} - 2v_n, v_n\}$. 