1. (15%) Find the points on the surface \( xy^2z^3 = 2 \) that are closest to the origin and also the shortest distance between the surface and the origin.

Solution:

Solution. Consider the Lagrange function \( F(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(xy^2z^3 - 2) \). The critical points of \( F(x, y, z, \lambda) \) satisfy

\[
\begin{align*}
F_x &= 2x - \lambda y^2 z^3 = 0 \\
F_y &= 2y - \lambda 2xyz^3 = 0 \\
F_z &= 2z - \lambda 3xy^2z^2 = 0 \\
F_{\lambda} &= -xy^2z^3 = 0.
\end{align*}
\]

(8pts)

To solve these equations, since \( x \neq 0, y \neq 0, z \neq 0 \) on the surface \( xy^2z^3 = 2 \), from (1), (2), and (3), we get

\[
\lambda = \frac{2x}{y^2z^3} = \frac{2y}{2xyz^3} = \frac{2z}{3xy^2z^2}.
\]

Second equality gives \( 2x^2 = y^2 \Rightarrow x = \pm \frac{\sqrt{2}}{2} y \). Third equality gives \( 2z^2 = 3y^2 \Rightarrow z = \pm \frac{\sqrt{2}}{3} y \).

We put \( x = \pm \frac{\sqrt{2}}{2} y \) and \( z = \pm \frac{\sqrt{2}}{3} y \) into (4) and get

\[
\left( \pm \frac{\sqrt{2}}{2} y \right)^2 \left( \pm \frac{\sqrt{6}}{2} y \right)^3 = 2 \Rightarrow y^6 = \left( \frac{2}{\sqrt{3}} \right)^3 \Rightarrow y^2 = \frac{2}{\sqrt{3}} \Rightarrow y = \pm \frac{\sqrt{2}}{\sqrt{3}}.
\]

(4pts)

So we get four critical points

\[
\left( \frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{3}}{3} \right), \quad \left( -\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, -\frac{\sqrt{3}}{3} \right), \quad \left( \frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{3}}{3} \right), \quad \left( -\frac{1}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}, -\frac{\sqrt{3}}{3} \right).
\]

(2pts)

These four critical points have the same distance to the origin:

\[
d = \sqrt{\left( \frac{1}{\sqrt{3}} \right)^2 + \left( \frac{\sqrt{2}}{\sqrt{3}} \right)^2 + \left( \frac{\sqrt{3}}{3} \right)^2} = \sqrt{\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{3}} = \sqrt{2\sqrt{3} - 2\frac{\sqrt{3}}{3}}.
\]

(1pts)

Solution 2. Another method to solve the system of equations (1) – (4) is comparing \( 6x \times (1), 3y \times (2), \) and \( 2z \times (3) \), then we get \( 12x^2 = 6y^2 = 4z^2 \). So we also find relations \( 2x^2 = y^2 \) and \( 2z^2 = 3y^2 \).

Solution 3. Instead of finding the maximum or minimum values of the function \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \), we consider its square function \( f^2(x, y, z) = x^2 + y^2 + z^2 \) because they both attain maximum or minimum at the same places.

Since \( xy^2z^3 = 1 \), we get \( y^2 = \frac{1}{xz^3} \), so the question reduce to find the absolute minimum of the following function of two variables:

\[
f(x, z) = x^2 + \frac{2}{xz^3} + z^2.
\]

The critical points of \( f(x, z) \) satisfy

\[
\begin{align*}
f_x &= 2x - \frac{2}{xz^3} = \frac{2x^3z^3 - 2}{xz^3} = \frac{2(x^3z^3 - 1)}{xz^3} = 0 \\
f_z &= -6 + \frac{6}{x} + 2z = 2z = \frac{2(-3 + x^4)}{x} = 0.
\end{align*}
\]

From \( f_x = 0 \), we get \( (xz)^3 = 1 \Rightarrow xz = 1 \). From \( f_z = 0 \), we get \( xz^5 = 3 \Rightarrow z^4 = 3 \Rightarrow z = \pm \sqrt[4]{3} \).
3. (12%) Let the unit vectors

(a) If \( z = \sqrt{3} \), then \( x = \frac{1}{\sqrt{3}} \), and \( y^2 = \frac{1}{x^2} = \frac{1}{\sqrt{3}} \implies y = \pm \frac{1}{\sqrt{3}}. \)

* At \( P_1 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \sqrt{3}) \), \( d(P_1, O) = \sqrt{2\sqrt{3}} = \sqrt{2}\sqrt{3}. \)
* At \( P_2 = (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \sqrt{3}) \), \( d(P_2, O) = \sqrt{2\sqrt{3}} = \sqrt{2}\sqrt{3}. \)

(b) If \( z = -\sqrt{3} \), then \( x = -\frac{1}{\sqrt{3}} \), and \( y^2 = \frac{1}{x^2} = \frac{1}{\sqrt{3}} \implies y = \pm \frac{1}{\sqrt{3}}. \)

* At \( P_3 = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\sqrt{3}) \), \( d(P_3, O) = \sqrt{2\sqrt{3}} = \sqrt{2}\sqrt{3}. \)
* At \( P_4 = (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\sqrt{3}) \), \( d(P_4, O) = \sqrt{2\sqrt{3}} = \sqrt{2}\sqrt{3}. \)

These four critical points have the shortest distance between the surface and the origin.

2. (12%) Find all the critical points of \( f(x, y) = 4 + x^3 + y^3 - 3xy \). Then determine which gives a local maximum or a local minimum or a saddle point.

Solution:

- \( f(x, y) = 4 + x^3 + y^3 - 3xy \implies (f_x, f_y) = (3x^2 - 3y, 3y^2 - 3x) \) (1% for \( f_x \), 1% for \( f_y \))

- Solve

\[
\begin{align*}
  f_x &= 3x^2 - 3y = 0 \\
  f_y &= 3y^2 - 3x = 0.
\end{align*}
\]

to obtain \((x, y) = (0, 0), (1, 1)\). Therefore the critical points are \((x, y) = (0, 0), (1, 1)\).

\((f_x = 0 : 1\%, f_y = 0 : 1\% \text{ solving: } 1\%)

- \((f_x, f_y) = (3x^2 - 3y, 3y^2 - 3x) \implies f_{xx} = 6x, f_{x} = f_{xy} = f_{yx} = -3, f_{yy} = 6y, \text{ and}

\[
D(x, y) = \left| \begin{array}{cc}
  f_{xx}(x, y) & f_{xy}(x, y) \\
  f_{yx}(x, y) & f_{yy}(x, y)
\end{array} \right| = \left| \begin{array}{cc}
  6x & -3 \\
  -3 & 6y
\end{array} \right| = 36xy - 9.
\]

\((f_{xx} = 1\% , f_{xy} = 1\% , f_{yx} = 1\% , f_{yy} = 1\% , D(x, y) = 1\%)

- \( D(0, 0) = -9 < 0 \implies (0, 0) \) is a saddle point. (1%)

- \( f_{xx}(1, 1) = 6 > 0 \) and \( D(1, 1) = 27 > 0 \implies f(1, 1) \) is a local minimum. (1%)

3. (12%) Let the unit vectors \( \mathbf{u} \) and \( \mathbf{n} \) be respectively the tangent direction and the normal direction (with positive \( x \)-component) of the circle \( x^2 + y^2 - 2x = 0 \) at the point \( \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \). Let \( f(x, y) = \tan^{-1} \left( \frac{y}{x} \right) \). Find \( \nabla f \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), D_uf \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \) and \( D_nf \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \).

Solution:

- \( f_x(x, y) = \frac{1}{1 + \frac{y^2}{x^2}} \frac{\partial f}{\partial x} \left( \frac{y}{x} \right) = \frac{-y}{x^2 + y^2} \). (2 points)

- \( f_y(x, y) = \frac{-x}{x^2 + y^2} \). (2 points)

- \( \nabla f(x, y) = \frac{-y}{x^2 + y^2} \hat{i} + \frac{x}{x^2 + y^2} \hat{j} \implies \nabla f \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) = \frac{\sqrt{3}}{2} \hat{i} + \frac{1}{2} \hat{j} \). (1 point)

- The equation \( x^2 + y^2 - 2x = 0 \) can be rewritten as \((x - 1)^2 + y^2 = 1\), which represents a circle centered at \((1, 0)\). Let \( F(x, y) = x^2 + y^2 - 2x \).

  - The normal direction of the circle at \((x, y)\) is \( \nabla F = (2x - 2, 2y) = 2(x - 1, y) \). (2 points)

  - The tangent direction of the circle at \((x, y)\) is \((y, -x + 1)\). (1 point)

  - At \( \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \), the normal direction with the positive \( x \)-component is \( \mathbf{n} = \left( \frac{1}{2}, -\frac{\sqrt{3}}{2} \right) \). (1 point)
− At \((\frac{1}{2}, \sqrt{3})\), the tangent direction with the positive \(x\)-component is \(\vec{u} = (\frac{\sqrt{3}}{2}, \frac{1}{2})\). (1 point)

• \(D f(\frac{1}{2}, \sqrt{3}) = (\frac{\sqrt{3}}{2}, \frac{1}{2}) \cdot (\frac{1}{2}, -\sqrt{3}) = -\frac{\sqrt{3}}{2}\). (1 point)

• \(D f(\frac{1}{2}, \sqrt{3}) = (\frac{\sqrt{3}}{2}, \frac{1}{2}) \cdot (\frac{\sqrt{3}}{2}, \frac{1}{2}) = -\frac{1}{2}\). (1 point)

4. (a) (10%) Find the 4-th degree MacLaurin polynomials of \(\sec x\) and of \((1 - x^2)^{-\frac{1}{2}}\) (5% each).

(b) (4%) Find \(\lim_{x \to 0} \frac{\sec x - (1 - x^2)^{-\frac{1}{4}}}{x^4}\).

Solution:

(a) The 4-th degree MacLaurin polynomial of \(\sec x\) can be derived from the cosine function: since

\[
\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots,
\]

the MacLaurin polynomial of \(\sec x\) can be obtained using long division or by comparing the coefficients in

\[
1 = (\cos x) \cdot (\sec x) = \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots\right) \cdot (a_0 + a_2 x^2 + a_4 x^4 + \cdots)
\]

Then we have

\[
\sec x = 1 + \frac{1}{2} x^2 + \frac{5}{24} x^4 + \cdots.
\]

(Since only a finite number of terms are required, you may also use the definition of MacLaurin polynomial: \(f(x) = f(0) + f'(0) x + \frac{f''(0)}{2!} x^2 + \cdots\) and perform the required differentiation to get the answer.)

On the other hand, by the binomial expansion

\[
(1 - x^2)^{-\frac{1}{4}} = \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right) (-x^2)^n
\]

\[
= 1 + \frac{1}{2} x^2 + \frac{3}{8} x^4 + \cdots
\]

Grading policy: 5 points for each polynomial. Three points are credited if only two terms are correct.

(b) From the results in part (a)

\[
\lim_{x \to 0} \frac{\sec x - (1 - x^2)^{-\frac{1}{4}}}{x^4} = \lim_{x \to 0} \frac{(1 + \frac{1}{2} x^2 + \frac{5}{24} x^4 + \cdots) - (1 + \frac{1}{2} x^2 + \frac{3}{8} x^4 + \cdots)}{x^4}
\]

\[
= \lim_{x \to 0} \frac{-\frac{1}{8} x^4 + \cdots}{x^4}
\]

\[
= -\frac{1}{6} \quad (4 \text{ points})
\]

5. (a) (12%) Find the radius of convergence and the interval of convergence of the power series \(f(x) = \sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n}\).

(b) (3%) Evaluate \(f^{(3)}(0)\).

Solution:

(a) By Ratio Test, \(f(x)\) converges absolutely if \(\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \) (4%)

\[
\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \left|\frac{(-1)^{n+1} x^{n+1}}{4^{n+1} \ln(n+1)} \cdot \frac{4^n \ln n}{x^n}\right| = \lim_{n \to \infty} \left|\frac{x \ln n}{4 \ln n + 1}\right| = \left|\frac{x}{4}\right| < 1 \quad (2%)
\]

Since \(\lim_{y \to \infty} \left|\frac{\ln y}{\ln y + 1} \cdot \frac{y + 1}{y}\right| = 1\)

Hence, the radius of convergence is 4.
for $x = 4$, $f(x) = \sum_{n=2}^{\infty} (-1)^n \frac{1}{n \ln n}$ converges.

Since (i) $\lim_{n \to \infty} \frac{1}{n \ln n} = 0$, (ii) $\frac{1}{n \ln n} > \frac{1}{n \ln n + 1}$ ⇒ converges by Alternating Series Test. (3%)

for $x = -4$, $f(x) = \sum_{n=2}^{\infty} \frac{1}{n \ln n} > \sum_{n=2}^{\infty} \frac{1}{n}$ diverges.

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ is p-series of $p = 1$ ⇒ diverges (3%)

(b) $f'(x) = \sum_{n=2}^{\infty} (-1)^n \frac{x^{n-1}}{4^n \ln n} \times n$

$f''(x) = \sum_{n=2}^{\infty} (-1)^n \frac{x^{n-2}}{4^n \ln n} \times (n(n-1))$ (1%)

$f^{(3)}(x) = \sum_{n=3}^{\infty} (-1)^n \frac{x^{n-3}}{4^n \ln n} \times (n(n-1)(n-2))$ (1%)

$f^{(3)}(0) = (-1)^3 \frac{1}{4^3 \ln 3} \times 3 \times 2 \times 1 = -\frac{6}{4^3 \ln 3}$ (1%)

< Solution2 > $f^{(3)}(0) = \frac{(-1)^3}{4^3 \ln 3}$ (1%)$f^{(3)}(0) = -\frac{6}{4^3 \ln 3}$

6. (10%) Suppose that $z = f(x, y)$ is a smooth function and let $x = uv$, and $y = v - u$. Express $\frac{\partial^2 z}{\partial u \partial v}$ in terms of $x, y, f_x, f_y, f_{xx}, f_{xy}$, and $f_{yy}$.

**Solution:**

The chain rule gives

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}$$

(3 points)

$$= \frac{\partial f}{\partial u} + \frac{\partial f}{\partial y}$$

(2 points)

$$= \frac{\partial f}{\partial u} + \frac{\partial f}{\partial y}$$

Apply the product rule and again the chain rule, and also note that $\frac{\partial}{\partial u}(u) = 1$,

$$\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial u} \left( \frac{u \cdot \partial f}{\partial x} + \frac{\partial f}{\partial y} \right)$$

$$= \frac{\partial f}{\partial x} + u \frac{\partial^2 f}{\partial u \partial x} + \frac{\partial^2 f}{\partial u \partial y}$$

$$= \frac{\partial f}{\partial x} + u \left( \frac{\partial^2 f}{\partial x^2} \frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial y \partial x} \frac{\partial u}{\partial y} \right) + \left( \frac{\partial^2 f}{\partial x \partial y} \frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial y^2} \frac{\partial u}{\partial y} \right)$$

$$= \frac{\partial f}{\partial x} + u \left( \frac{\partial^2 f}{\partial x^2} v + \frac{\partial^2 f}{\partial y \partial x} (-1) \right) + \left( \frac{\partial^2 f}{\partial x \partial y} v + \frac{\partial^2 f}{\partial y^2} (-1) \right)$$

$$= \frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x^2} v + \frac{\partial^2 f}{\partial y \partial x} (-1) - \frac{\partial^2 f}{\partial y^2} (-1)$$

Since $f$ is a smooth function, $f_{xy} = f_{yx}$. Therefore

$$\frac{\partial^2 z}{\partial u \partial v} = uv f_{xx} + (v - u) f_{xy} - f_{yy} + f_x$$

(4 points)

$$= x f_{xx} + y f_{xy} - f_{yy} + f_x$$

(1 point)

(Because $f$ satisfies the condition of Clairaut’s theorem, you can first calculate $z_u$ and $z_{uv}$, and then claim that $z_{uv} = z_{vu}$. This approach yields the same solution as above.)

7. (10%) Find the equation of the tangent plane to the elliptic paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ at the point $(a, b, 2c)$. 
Solution:
Let \( f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c} \)
then \( f_x = \frac{2x}{a^2}, f_y = \frac{2y}{b^2}, f_z = -\frac{1}{c} \)
at \((a, b, 2c)\), we have \( f_x = \frac{2a}{a^2}, f_y = \frac{2b}{b^2}, f_z = -\frac{1}{c} \)
Hence the tangent plane is \( \frac{2}{a}(x - a) + \frac{2}{b}(y - b) - \frac{1}{c}(x - 2c) = 0 \)

\[
\begin{align*}
\text{Solution:} \\
\text{Let } f(x, y, z) &= \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c} \\
\] then \( f_x = \frac{2x}{a^2}, f_y = \frac{2y}{b^2}, f_z = -\frac{1}{c} \) at \((a, b, 2c)\), we have \( f_x = \frac{2a}{a^2}, f_y = \frac{2b}{b^2}, f_z = -\frac{1}{c} \) Hence the tangent plane is \( \frac{2}{a}(x - a) + \frac{2}{b}(y - b) - \frac{1}{c}(x - 2c) = 0 \)

8. (a) (2%) Parametrize the curve of intersection of the parabolic cylinders \( x = y^2 \) and \( z = x^2 \) by setting \( t = y \).
(b) (10%) Find the unit tangent \( T \) and the curvature \( \kappa \) at the point \((1, 1, 1)\).

Solution:
(a) Since \( x = y^2, z = x^2 \), and \( y = t \), we can see \( x(t) = t^2 \) and \( z(t) = t^4 \).
\[
\begin{align*}
x(t) &= t^2 \\
y(t) &= t \\
z(t) &= t^4 \\
\end{align*}
\]
or write as \( r(t) = (t^2, t, t^4), t \in \mathbb{R} \).
Although I do not deduction any points, you should still write down range of \( t \).

(b) By formula \( T(t) = \frac{r'(t)}{|r'(t)|} \), and easy to know \( r'(t) = (2t, 1, 4t^3) \), so \( r'(1) = (2, 1, 4) \), and \( |r'(t)| = \sqrt{21} \). So we get \( T(1) = \frac{(2, 1, 4)}{\sqrt{21}} \).
If you do perfect, you get 5 points. If you compute some error, you will get from 1 to 4 points, depending your answer. If you use wrong formula, you will get 0 or 1 point.

By formula \( \kappa(t) = \frac{|r''(t) \times r'(t)|}{|r'(t)|^3} \),

First, \( r''(t) = (2, 0, 12t^2) \), so \( r'(1) \times r''(1) = (2, 1, 4) \times (2, 0, 12) = (12, -16, -2) \). So \( |r'(1) \times r''(1)| = \sqrt{404} = 2\sqrt{101} \), and \( |r'(1)|^3 = 21\sqrt{21} \). We get the answer is \( \kappa(1) = \frac{2\sqrt{101}}{21\sqrt{21}} \).
If you do perfect, you get 5 points. If you compute some error, you will get from 1 to 4 points, depending your answer. If you use wrong formula, you will get 0 or 1 point.