1. (5%) Determine the statement is true (✓) or false (✗).

(a) If \( f(x, y) \) is continuous on the rectangle \( R = \{(x, y) \mid a \leq x \leq b, \ c \leq y \leq d\} \) except for finitely many points, then \( f(x, y) \) is integrable on \( R \) and

\[
\iint_R f(x, y) \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy = \int_a^b \int_c^d f(x, y) \, dy \, dx.
\]

(b) If \( \mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j} \) and \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \) on an open connected region \( D \), then \( \mathbf{F} \) is conservative on \( D \).

(c) If \( \text{curl} \, \mathbf{F} = \text{curl} \, \mathbf{G} \) on \( \mathbb{R}^3 \), then \( \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{G} \cdot d\mathbf{r} \) for all closed path \( C \).

(d) If \( \mathbf{F} \) and \( \mathbf{G} \) are vector fields and \( \text{curl} \, \mathbf{F} = \text{curl} \, \mathbf{G} \), \( \text{div} \, \mathbf{F} = \text{div} \, \mathbf{G} \), then \( \mathbf{F} - \mathbf{G} \) is a constant vector field.

(e) Let \( B \) be a rigid body rotating about the \( z \)-axis with constant angular speed \( \omega \). If \( \mathbf{v}(x, y, z) \) is the velocity at point \( (x, y, z) \in B \), then \( \text{curl} \, \mathbf{v} \) is parallel to \( \mathbf{k} \).

Answer. (每小題各 1 分)

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2. (10%) Write the integral \( \int_0^1 \int_0^{\sqrt{x}} \int_0^{1-y} f(x, y, z) \, dz \, dy \, dx \) in 5 other orders.

Answer. (每小題一格扣 1 分，錯兩格以上全錯)

(a) \( \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x, y, z) \, dz \, dx \, dy \)

(b) \( \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x, y, z) \, dx \, dz \)

(c) \( \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x, y, z) \, dx \, dz \)

(d) \( \int_0^1 \int_0^{1-\sqrt{x}} \int_0^{1-z} f(x, y, z) \, dy \, dx \)

(e) \( \int_0^1 \int_0^{(1-z)^2} \int_0^{1-z} f(x, y, z) \, dy \, dz \)
3. (15%) Evaluate the integrals.

(a) \[
\int_0^1 \int_{\tan^{-1} y}^{\pi/4} \cos x \tan (\cos x) \, dx \, dy.
\]

(b) \[
\int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x + y}{x^2 + y^2} \, dy \, dx + \int_1^{1-y} \int_{x^2 + y^2}^{x + y} \frac{x + y}{x^2 + y^2} \, dy \, dx + \int_0^{\sqrt{2}} \int_{x^2 + y^2}^{\sqrt{2-x^2}} \frac{x + y}{x^2 + y^2} \, dy \, dx.
\]

Solution:

(a)
\[
\int_0^1 \int_{\tan^{-1} y}^{\pi/4} \cos x \tan (\cos x) \, dx \, dy = \int_0^{\pi/4} \tan x \cos x \tan (\cos x) \, dx \quad (3pt)
\]
\[
= \int_0^{\pi/4} \sin x \tan (\cos x) \, dx \quad \text{(Let } u = \cos x, \, du = -\sin x \, dx) \quad (2pt)
\]
\[
= -\int_0^{\pi/4} \tan u \, du \quad (1pt)
\]
\[
= \ln(\cos u)|_0^{\pi/4} \quad (2pt)
\]
\[
= \ln(\cos 1) - \ln(\cos 1) \quad (1pt)
\]

(b)
\[
\int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x + y}{x^2 + y^2} \, dy \, dx + \int_1^{1-y} \int_{x^2 + y^2}^{x + y} \frac{x + y}{x^2 + y^2} \, dy \, dx + \int_0^{\sqrt{2}} \int_{x^2 + y^2}^{\sqrt{2-x^2}} \frac{x + y}{x^2 + y^2} \, dy \, dx = \iint_D \frac{x + y}{x^2 + y^2} \, dA, \quad \text{where } D \text{ is bounded by } x^2 + y^2 = 2 \text{ and } x + y = 1.
\]

By polar coordinate, we have
\[
\iint_D \frac{x + y}{x^2 + y^2} \, dA = \int_0^{\pi/2} \int_{\cos \theta + \sin \theta}^{\sqrt{2}} \frac{r \cos \theta + r \sin \theta}{r^2} \cdot r \, dr \, d\theta \quad (4pt)
\]
\[
= \int_0^{\pi/2} \int_{\cos \theta + \sin \theta}^{\sqrt{2}} (\cos \theta + \sin \theta) \, dr \, d\theta \quad (3pt)
\]
\[
= \sqrt{2}(\sin \theta - \cos \theta)|_0^{\pi/2} - \frac{\pi}{2} \quad (3pt)
\]
\[
= 2\sqrt{2} - \frac{\pi}{2} \quad (2pt)
\]
4. (10%) Let $S$ be the surface $x^2 + y^2 + z^2 = a^2$, $x \geq 0$, $y \geq 0$, $z \geq 0$ $(a > 0)$, and let $C$ be the boundary of $S$. Find the centroid of $C$.

**Solution:**

For a quarter circle of radius $a$ (named $C'$) on a plane, its centroid can be found to be at $(\frac{2a}{\pi}, \frac{2a}{\pi})$ by either way:

(1) Parametrize the curve:

Parametrize $C'$ by $r(t) = (a \cos t, a \sin t), t \in [0, \frac{\pi}{2}]$.

$\Rightarrow |r'(t)| = [-a \sin t, a \cos t] = a$.

Arc length $s = \frac{1}{4} \cdot 2\pi a = \frac{1}{2} \pi a$.

$x \cdot s = \int_{C'} x \cdot ds = \int_0^{\pi/2} x(t)|r'(t)|dt = \int_0^{\pi/2} a \cos(t)adt = a^2$

$\therefore x = \frac{a^2}{\frac{1}{2} \pi a} = \frac{2a}{\pi}$

By the symmetry of the arc, $y = x = \frac{2a}{\pi}$.

(2) Pappus’s Theorem:

Knowing that the surface area of a hemisphere of radius $a$ is $2\pi a^2$ and the arc length of a quarter circle of radius $a$ is $\frac{1}{2} \pi a$, if the quarter circle is in the first quadrant and is rotated about the $x$-axis, Pappus’s Theorem gives

$A = 2\pi y \cdot s$

$\Rightarrow 2\pi a^2 = 2\pi y \cdot \frac{\pi a}{2}$

$\Rightarrow y = \frac{2a}{\pi}$

By the symmetry of the arc, $y = x = \frac{2a}{\pi}$.

(7 points up to this point.)

The curve $C$ is composed of quarter circles $C_1$, $C_2$, and $C_3$ on the $xy$-, $yz$-, and $xz$-planes, respectively. By the above discussion, their centroids are $(\frac{2a}{\pi}, \frac{2a}{\pi}, 0)$, $(0, \frac{2a}{\pi}, \frac{2a}{\pi})$, and $(\frac{2a}{\pi}, 0, \frac{2a}{\pi})$, respectively. Since they have equal masses, the centroid of $C$ is the average of them, namely $(\frac{4a}{3\pi}, \frac{4a}{3\pi}, \frac{4a}{3\pi})$. (3 points)

(Note: if you misunderstood the problem but correctly calculated the centroid of the surface $S$ to be at $(\frac{a}{\pi}, \frac{a}{\pi}, \frac{a}{\pi})$, you still get 4 points. But no points will be given if you calculated the centroid of the part of the volume inside the sphere in the first octant.)
5. (10%) Let $C$ be the curve of intersection of \(x^2 + y^2 + z^2 = 4\), \(x^2 + y^2 = 2x\), \(z \geq 0\), oriented $C$ to be counterclockwise when viewed from above. Evaluate $\int_C y^2 dx + z^2 dy + x^2 dz$.

Solution:

- **Solution 1**: Using line integral to solve this problem directly.
  
  \[
  r(\theta) = <1 + \cos \theta, \sin \theta, 2 \sin \frac{\theta}{2}> \quad \theta \in [0, 2\pi].
  \]
  
  The original equation $= \int_0^{2\pi} \left[- \sin^3 \theta + \sin^2 \frac{\theta}{2} \cos \theta + (1 + \cos \theta)^2 \cos \frac{\theta}{2}\right] d\theta$ (3 points)
  
  By symmetry, the first and the third term will be zero in the end.
  
  Therefore, the above equation will change as follows:
  
  $4\int_0^{2\pi} \frac{1}{2} \cos^2 \theta \cos \theta \, d\theta$
  
  $= -2\pi$ (4 points)

- **Solution 2**: Using Stokes’ Theorem to solve this problem.
  
  $F = <y^2, z^2, x^2>$
  
  $\nabla \times F = <-2 z, -2 x, -2 y>$ (2 points)
  
  $r(x, y) = <x, y, \sqrt{4 - x^2 - y^2}>$
  
  $r_x = <1, 0, \frac{x}{\sqrt{4 - x^2 - y^2}}>$
  
  $r_y = <0, 1, \frac{-y}{\sqrt{4 - x^2 - y^2}}>$
  
  $r_x \times r_y = <\frac{y}{\sqrt{4 - x^2 - y^2}}, \frac{x}{\sqrt{4 - x^2 - y^2}}, 1>$ (2 points)
  
  By Stokes’ Theorem,
  
  $\oint_C F \cdot \, dr = \iint_D (\nabla \times \, F) \cdot \, dS$
  
  $= \iint_D \left(-2 \frac{2y}{\sqrt{4-x^2-y^2}}, -2x, -2y\right) \, dA$
  
  By symmetry, the second and the third term will be zero in the end.
  
  Therefore, the above equation will change as follows:
  
  $= -2 \int_0^{2\pi} \left(\frac{1}{2} \int_0^{\frac{\pi}{2}} \cos \theta r \cos \theta \, r \, dr \, d\theta\right)$
  
  $= -2 \int_0^{2\pi} \frac{1}{4} \sin \theta \, d\theta$
  
  $= -\frac{\pi}{2}$ (3 points)
6. (20%) Let \( \mathbf{F} = \frac{(x-y)^2}{(x+y)^2} \mathbf{i} + \frac{(x-y)^2}{(x+y)^2} y \mathbf{j} \).

(a) Verify that \( \mathbf{F} \) is conservative on the right half plane \( x > 0 \). Find a potential function of \( \mathbf{F} \) on the right half plane.

(b) Evaluate \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} \) where \( C_1 \) is the ellipse \( \frac{x^2}{4} + (y-2)^2 = 1 \).

(c) Evaluate \( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \) where \( C_2 \) is the curve with polar equation \( r = e^{|\theta|}, \ -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \).

![Diagram](Diagram.png)

\[ \text{Solution:} \]

(a) (6%) \( \partial_y f = \frac{(x-y)^2}{(x+y)^2} \Rightarrow f = \frac{-x^2}{(x+y)^2} \tan^{-1}(\frac{y}{x}) + g(x) \)

\( \Rightarrow \partial_x f = \frac{-2xy}{(x+y)^2} + \frac{y}{(x+y)^2} + g'(x) = \frac{(x-y)^2}{(x+y)^2} + g'(x) = \frac{(x-y)^2}{(x+y)^2} \tan^{-1}(\frac{y}{x}) \)

\( \Rightarrow g'(x) = 0 \Rightarrow g \) is constant

\( \Rightarrow f = \frac{-x^2}{(x+y)^2} \tan^{-1}(\frac{y}{x}) + 1 + \frac{-y^2}{(x+y)^2} \tan^{-1}(\frac{y}{x}) = \frac{-y^2}{(x+y)^2} - \tan^{-1}(\frac{y}{x}) \) (6%)

Other point: \( P_y = Q_x = \frac{x^2 - 4y^2 - 4xy - y^4}{(x+y)^2} \) (2%); " \( P_y = Q_x \) " implies \( f \) is conservative (1%)

(b) (4%) Since \( \{y > 0\} \) is simple connected, \( \mathbf{F} \) is conservative on \( \{y > 0\} \).

On the other hand, \( C_1 \) is closed curve on \( y > 0 \) (1%); therefore, \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0 \) (3%)

(c) (10%) "method 1"

the integral on \( C_2 \) is equal to the integral on unit circle times two and integral on the tail.

\( \int_{D} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{2\pi} -(\cos \theta - \sin \theta)^2 d\theta = -2\pi \) (4%), where \( D \) is unit circle.

The integral on tail is independent of path, which equals to

\( f(\cos \frac{2\pi}{4} e^{\frac{\pi i}{4}}, \sin \frac{2\pi}{4} e^{\frac{\pi i}{4}}) - f(\cos \frac{2\pi}{4} e^{\frac{\pi i}{4}}, \sin \frac{-2\pi}{4} e^{\frac{-\pi i}{4}}) = \frac{-\pi}{2} \) (2%), where \( f \) is potential function of \( \mathbf{F} \)

Therefore, \( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -4\pi - \frac{\pi}{2} = -\frac{9\pi}{2} \) (2%)

"method 2"

\( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{\gamma_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\gamma_2} \mathbf{F} \cdot d\mathbf{r} \), where \( \gamma_1 \) is the " \( \theta \geq 0 \) " part of \( C_2 \), \( \gamma_2 \) is " \( \theta < 0 \) " part of \( C_2 \), in which \( x(\theta) \) and \( y(\theta) \) is differentiable.

\( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{\pi} -(\cos \theta - \sin \theta)^2 d\theta + \int_{-\pi}^{0} -(\cos \theta - \sin \theta)^2 d\theta \) (4%)

\( = \int_{-\pi}^{\pi} -(\cos \theta - \sin \theta)^2 d\theta = -\frac{9\pi}{2} \) (6%) (the answer worth 2 point)
7. (10%) Evaluate \[\int_C (y + \sin^3 x) \, dx + (z^2 + \cos^4 y) \, dy + (x^3 + \tan^5 z) \, dz\] where \(C\) is the curve \(r(t) = \sin t \, \mathbf{i} + \cos t \, \mathbf{j} + \sin 2t \, \mathbf{k}\), \(0 \leq t \leq 2\pi\). [Hint: \(C\) lies on the surface \(z = 2xy\).]

**Solution:**
First observe that \(r(t) = \sin t \, \mathbf{i} + \cos t \, \mathbf{j} + \sin 2t \, \mathbf{k}\) is negative oriented. Thus by Stoke’s theorem:
\[
\int_C (y + \sin^3 x) \, dx + (z^2 + \cos^4 y) \, dy + (x^3 + \tan^5 z) \, dz = -\int_S \nabla F \cdot dS
\]
where \(S\) is the surface \(z = 2xy\) bounded by \(D = \{(x^2 + y^2 \leq 1)\}
\[
\nabla F = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
y + \sin^3 x & z^2 + \cos^4 y & x^3 + \tan^5 z
\end{vmatrix} = -2z \mathbf{i} - 3x^2 \mathbf{j} - \mathbf{k}
\]
\[
\mathbf{n} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) / \left(\left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right)\right) = (-2y, -2x, 1) / \left((-2y, -2x, 1)\right)
\]
\[
\int_S \nabla F \cdot dS = \iint_D (-2z, -3x^2, -1) \cdot (-2y, -2x, 1) \, dA
= \iint_D 4yz + 6x^3 - 1 \, dA
= \iint_D 8x^2y + 6x^3 - 1 \, dA
= \iiint_0^{2\pi} \int_0^1 (8r^3 \cos \theta \sin^2 \theta + 6r^3 \cos^3 \theta - 1) r \, dr \, d\theta
= -\pi
\]
\[
\int_C (y + \sin^3 x) \, dx + (z^2 + \cos^4 y) \, dy + (x^3 + \tan^5 z) \, dz = \pi
\]
8. (10%) Evaluate \( \iiint_S x \, dS \) where \( S \) is the part of the cone \( z = \sqrt{2(x^2 + y^2)} \) that lies below the plane \( z = 1 + x \).

**Solution:**

**Step 1.**

Find the projection onto the \( xy \)-plane of the curve of intersection of the cone \( z = \sqrt{2(x^2 + y^2)} \) and the plane \( z = 1 + x \).

\[
\begin{align*}
&\begin{cases}
  z = \sqrt{2(x^2 + y^2)} \\
  z = 1 + x
\end{cases} \\
\Rightarrow & 2(x^2 + y^2) = (x + 1)^2 \\
\Rightarrow & \left( \frac{x - 1}{\sqrt{2}} \right)^2 + y^2 = 1 \quad (1pt)
\end{align*}
\]

**Step 2.**

If we regard \( x \) and \( y \) as parameters, then we can write the parametric equations of \( S \) as

\[
\begin{align*}
  x &= x \\
  y &= y \\
  z &= \sqrt{2(x^2 + y^2)}
\end{align*}
\]

where

\[
1 - \sqrt{2(1 - y^2)} \leq x \leq 1 + \sqrt{2(1 - y^2)} , \quad -1 \leq y \leq 1 \quad (1pt)
\]

and the vector equation is

\[
r(x, y) = xi + yj + \sqrt{2(x^2 + y^2)}k
\]

**Step 3.**

Find \( |r_x \times r_y| \).

\[
\begin{align*}
  r_x &= li + 0j + \frac{\sqrt{2x}}{\sqrt{x^2 + y^2}} k \\
  r_y &= 0i + lj + \frac{\sqrt{2y}}{\sqrt{x^2 + y^2}} k
\end{align*}
\]

\[
\Rightarrow r_x \times r_y = \frac{-\sqrt{2x}}{\sqrt{x^2 + y^2}} i + \frac{-\sqrt{2y}}{\sqrt{x^2 + y^2}} j + 1k \quad (2pts)
\]

\[
\Rightarrow |r_x \times r_y| = \sqrt{3} \quad (1pt)
\]

**Step 4.**

Evaluate \( \iiint_S x \, dS \).

\[
\begin{align*}
\iiint_S x \, dS &= \iint_D x \cdot |r_x \times r_y| \, dx \, dy \quad (2pts) \\
&= \sqrt{3} \int_{-1}^{1} \int_{1 - \sqrt{2(1 - y^2)}}^{1 + \sqrt{2(1 - y^2)}} x \, dx \, dy \\
&= \sqrt{3} \cdot 2\sqrt{2} \int_{-1}^{1} \sqrt{1 - y^2} \, dy \\
&= 2\sqrt{3} \cdot \frac{\pi}{2} \\
&= \sqrt{6}\pi \quad (2pts)
\end{align*}
\]
9. (10%) Let \( S \) be the surface of the solid bounded by \( x^2 + y^2 + z^2 = 1 \) and \( z \geq \frac{1}{2} \). Find the total flux of \( \mathbf{F}(x, y, z) = x^2 \mathbf{i} + y^2 \mathbf{j} + z^2 \mathbf{k} \) across \( S \).

Solution:

(Method I)

Let \( V = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq 1, \ z \geq \frac{1}{2}\} \), then by Divergence Theorem,

\[
\text{Flux of } \mathbf{F} = \iint_S \mathbf{F} \cdot dS = \iiint_V \text{div} \mathbf{F} \ dV = \iiint_V 2(x + y + z) \ dV \quad (3\%)
\]

From the symmetry of \( V \), we have \( \iiint_V x \ dS = \iiint_V y \ dS = 0 \).

Therefore, \( \iiint_V \text{div} \mathbf{F} \ dS = \iiint_V 2z \ dS = 2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_{\frac{1}{2}}^{1} \rho \cos \phi \rho^2 \sin \phi \ d\rho \ d\phi \ d\theta = \frac{9\pi}{64} \quad (7\%) \)

(Method II)

Let \( S_1 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1, \ z \geq \frac{1}{2}\} \) and \( S_2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 \leq \frac{1}{4}, \ z = \frac{1}{2}\} \)

\[
\text{Flux of } \mathbf{F} = \iint_S \mathbf{F} \cdot dS = \iint_{S_1} (x^2, y^2, z^2) \cdot dS + \iint_{S_2} (x^2, y^2, z^2) \cdot dS
\]

\[
= \iint_{S_1} (x^2, y^2, z^2) \cdot (x, y, z) \ dS + \iint_{S_2} (x^2, y^2, z^2) \cdot (0, 0, -1) \ dS.
\]

\[
= \iint_{S_1} x^3 + y^3 + z^3 \ dS - \iint_{S_2} z^2 \ dS = \iint_{S_1} x^3 + y^3 + z^3 \ dS - \frac{1}{4} \text{Area}(S_2). \quad (3\%)
\]

From the symmetry of \( S_1 \), we have \( \iint_{S_1} x^3 dS = \iint_{S_1} y^3 dS = 0 \).

Thus, we only need to calculate \( \iint_{S_1} z^3 dS \), then by Spherical coordinate

\[
\iint_{S_1} z^3 \ dS = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos \phi \rho^3 \sin \phi \ d\rho \ d\phi \ d\theta = 2\pi \cdot \left( -\frac{1}{4} \cos^4 \phi \right) \bigg|_0^{\frac{\pi}{2}} = \frac{\pi}{2} \left( \frac{1}{2} \right)^4 - 1 = \frac{15\pi}{32}.
\]

\[
\Rightarrow \iint_{S_1} \mathbf{F} \cdot dS = \iint_{S_1} z^3 dS - \frac{1}{4} \text{Area}(S_2) = \frac{15\pi}{32} - \frac{1}{4} \left( \frac{\sqrt{3}}{2} \right)^2 \pi = \frac{9\pi}{32} \quad (7\%)
\]
10. (10%) Solve the differential equation

\[ y'' + y = x^2 e^x + \tan x, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \]

**Solution:**

Complementary equation: \( y'' + y = 0. \)

Auxiliary equation: \( r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y_c = c_1 \sin x + c_2 \cos x. \) (2 points)

For the particular solution:

(1) For \( y'' + y = x^2 e^x \) it’s a better idea to use the method of undetermined coefficients:

Let \( y_{p_1} = (Ax^2 + Bx + C)e^x, \)

\[ y_{p_1}'' = [Ax^2 + (2A + B)x + (B + C)]e^x, \]

\[ y_{p_1}'' + y_{p_1} = [2Ax^2 + (4A + 2B)x + (2A + 2B + 2C)]e^x \equiv x^2 e^x \]

\[ \Rightarrow A = \frac{1}{2}, B = -1, C = \frac{1}{2} \]

\[ \therefore y_{p_1} = \left(\frac{1}{2}x^2 - x + \frac{1}{2}\right)e^x. \]

(2 points for the formulation, 2 points for solving the coefficients.)

(2) For \( y'' + y = \tan x \) we use the method of variation of parameters:

Let \( y_{p_2} = u_1 \sin x + u_2 \cos x, \)

\[ y_{p_2}' = (u_1' \sin x + u_2' \cos x) + u_1 \cos x - u_2 \sin x. \]

Setting \( u_1' \sin x + u_2' \cos x = 0 \) (equation 1), we have \( y_{p_2}'' = u_1' \cos x - u_2' \sin x - u_1 \sin x - u_2 \cos x. \)

\[ \Rightarrow y_{p_2}'' + y_{p_2} = u_1' \cos x - u_2' \sin x \equiv \tan x \] (equation 2).

Solving the system of equations 1 and 2, we have

\[
\begin{align*}
    u_1' \sin x + u_2' \cos x &= 0 \\
    u_1' \cos x - u_2' \sin x &= \tan x
\end{align*}
\]

\[ \Rightarrow \begin{cases} 
    u_1'(x) = \sin x, \\
    u_2'(x) = -\tan x \sin x = -\frac{\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x
\end{cases} \]

\[ \Rightarrow \begin{cases} 
    u_1(x) = -\cos x, \\
    u_2(x) = \sin x - \ln|\sec x + \tan x| = \sin x - \ln(\sec x + \tan x) \text{ for } x \in (-\frac{\pi}{2}, -\frac{\pi}{2})
\end{cases} \]

\[ \Rightarrow y_{p_2}(x) = u_1 \sin x + u_2 \cos x = -(\cos x) \ln(\sec x + \tan x) \]

(2 points for the system of equations, 2 points for solving and integrating them.)

Combining the above results, we have the general solution

\[ y(x) = c_1 \sin x + c_2 \cos x + \left(\frac{1}{2}x^2 - x + \frac{1}{2}\right)e^x - (\cos x) \ln(\sec x + \tan x) \]