1. (10%) Evaluate \( \int_0^2 \int_{x^3}^8 \frac{x^5}{\sqrt{x^6 + y^2}} dy dx \).

**Solution:**
Interchange the order of the iterated integral, we have
\[
\int_0^2 \int_{x^3}^8 \frac{x^5}{\sqrt{x^6 + y^2}} dy dx = \int_0^8 \int_0^{\sqrt[3]{y}} \frac{x^5}{\sqrt{x^6 + y^2}} dxdy \quad (3%)
\]
\[
= \int_0^8 \frac{1}{6} 2\sqrt{x^6 + y^2} \bigg|_0^{\sqrt[3]{y}} dy \quad (3%)
\]
\[
= \frac{1}{3} \int_0^8 (\sqrt[3]{2} - 1)y dy \quad (2%)
\]
\[
= \frac{1}{3}(\sqrt[3]{2} - 1)\frac{1}{2} y^2 \bigg|_0^8 \quad (1%)
\]
\[
= \frac{32}{3}(\sqrt[3]{2} - 1) \quad (1%)
\]
2. (15%) Find the volume of the solid common to the balls $\rho \leq 2\sqrt{2} \cos \phi$ and $\rho \leq 2$.

Solution:

Method 1.
Sphere $\rho = 2$ is equivalent to $x^2 + y^2 + z^2 = 4$; sphere $\rho = 2\sqrt{2} \cos \phi$ is equivalent to $x^2 + y^2 + (z - \sqrt{2})^2 = 2$. These two spheres intersect at $\phi = \pi/4$.

Therefore,

$$
\text{Volume} = \int_0^{2\pi} d\theta \int_0^{\pi/4} d\phi \int_0^{\sqrt{2}} \rho^2 \sin \phi d\phi + \int_0^{2\pi} d\theta \int_{\pi/4}^{\pi/2} d\phi \int_0^{2\sqrt{2} \cos \phi} \rho^2 \sin \phi d\phi
$$

$$
= 2\pi (-\cos \phi) \bigg|_0^{\pi/4} \cdot \left(\frac{\rho^3}{3}\right) \bigg|_0^{\sqrt{2}} + 2\pi \int_{\pi/4}^{\pi/2} \sin \phi \cdot \frac{1}{3} (2\sqrt{2} \cos \phi)^3 d\phi
$$

$$
= \frac{2\pi}{3} (8 - 4\sqrt{2}) + \frac{32\sqrt{2}\pi}{3} \int_{\pi/4}^{\pi/2} \sin \phi \cos^3 \phi d\phi
$$

$$
= \frac{2\pi}{3} (8 - 4\sqrt{2}) + \frac{2\pi}{3}
$$

$$
= \frac{16}{3} \pi - 2\sqrt{2}\pi.
$$

Note:
- $\theta : 0 \sim 2\pi$, (2 points), Jacobi factor $\rho^2 \sin \phi$, (5 points)

For part I, $\phi : 0 \sim \pi/4$, (2 points), $\rho : 0 \sim 2$, (1 point), answer $\frac{2\pi}{3} (8 - 4\sqrt{2})$, (1 point)

For part II, $\phi : \pi/4 \sim \pi/2$, (2 points), $\rho : 0 \sim 2\sqrt{2} \cos \phi$, (1 point), answer $\frac{2\pi}{3}$, (1 point)

Method 2.

$$
\text{Volume} = \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} dr \int_{\sqrt{4-r^2}}^{\sqrt{4-r^2}-\sqrt{2}} r dz
$$

$$
= 2\pi \int_0^{\sqrt{2}} r(\sqrt{4 - r^2} + \sqrt{2 - r^2} - \sqrt{2}) dr
$$

$$
= \frac{16}{3} \pi - 2\sqrt{2}\pi.
$$
Note:
\( \theta : 0 \sim 2\pi, \) (2 points)
\( z : \sqrt{2} - \sqrt{2 - r^2} \sim \sqrt{4 - r^2}, \) (3 points)
\( r : 0 \sim \sqrt{2}, \) (3 points)

Jacobi factor \( r, \) (5 points), answer \( \frac{2\pi}{3}(8 - 3\sqrt{2}), \) (2 points).

Method 3.

\[
\text{Volume} = \int_{\sqrt{2}}^{2} \pi(4 - z^2)dz + \frac{4}{3} \pi (\sqrt{2})^3 \cdot \frac{1}{2} II
\]
\[
= \frac{16}{3} \pi - 2\sqrt{2}\pi.
\]
3. (15%) Evaluate the integral \( \int_{\Omega} \sin(3x^2 - 2xy + 3y^2) \, dxdy \), where \( \Omega \) is the ellipse \( 3x^2 - 2xy + 3y^2 \leq 2 \). You may try the change of variables \( x = u + kv, \ y = u - kv \) for some constant \( k \).

**Solution:**

Follow the hint, set \( x = u + kv, \ y = u - kv \). Put in equation of the ellipse: (3 pts)

\[
3x^2 - 2xy + 3y^2 = 4u^2 + 8k^2v^2 \leq 2
\]

We can choose \( k = \frac{1}{\sqrt{2}} \), then the equation becomes \( 4u^2 + 4v^2 \leq 2 \). Calculate the Jacobian (value of \( J \) 2 pts, absolute value 2 pts):

\[
|J(u, v)| = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \begin{array}{cc} 1 & k \\ 1 & -k \end{array} \right| = | -2k | = \sqrt{2}
\]

Then

\[
\int_{\Omega} \sin(3x^2 - 2xy + 3y^2) \, dxdy
\]

\[= \int_{u^2+v^2 \leq \frac{1}{2}} \sin 4(u^2 + v^2)\sqrt{2} \, dudv \quad \text{(integrand 1 pt, domain 2 pts)}
\]

\[= \sqrt{2} \int_{0}^{2\pi} \int_{0}^{\frac{1}{\sqrt{2}}} \sin 4r^2 \, r \, dr \, d\theta \quad \text{(Jacobian of polar coordinates 3 pts)}
\]

\[= 2\sqrt{2}\pi \int_{0}^{\frac{1}{\sqrt{2}}} \frac{1}{8} \sin 4r^2 \, d(4r^2)
\]

\[= \frac{\sqrt{2}\pi}{4} (1 - \cos 2) \quad \text{(2 pts. If make a slight mistake, get 1 pt)}
\]

**Scoring to steps of this problem:**

1. Change of variable in \( u, v \): get 2 pt if the relation is correct.
2. Jacobian of your variable: get 2 pts for the value and 2 pts for absolute value.
3. Write down the correct integrand for your new variables: get 1 pt.
5. Change \( u, v \) into polar coordinates. If the Jacobian is correct: get 3 pts.
6. Your result fits the correct answer: get 2pts, and get 1 pt if you just make a slight mistake.
4. (15%) For \( y > 0 \), let
\[
\mathbf{F}(x,y,z) = (e^{-x \ln y - z}) \mathbf{i} + (2yz - e^{-x/y}) \mathbf{j} + (y^2 - x) \mathbf{k}
\] and
\[
\mathbf{G}(x,y,z) = e^{-x \ln y} \mathbf{i} + (2yz - e^{-x/y}) \mathbf{j} - x \mathbf{k}.
\]
(a) Show that the vector function \( \mathbf{F} \) is a gradient on \( \{(x,y,z) \mid y > 0\} \) by finding an \( f \) such that \( \nabla f = \mathbf{F} \).
(b) Evaluate the line integral \( \int_C \mathbf{G}(\mathbf{r}) \cdot d\mathbf{r} \), where \( C \) is the curve given by \( \mathbf{r}(u) = (1 + u^2) \mathbf{i} + e^u \mathbf{j} + (1 + u) \mathbf{k} \), \( u \in [0,1] \).

Solution:
(a) \( f = -e^{-x \ln y + y^2 z - xz + c} \), where \( c \in \mathbb{R} \) (5 points)

(b) \( \mathbf{G} = \mathbf{F} + z \mathbf{i} - y^2 \mathbf{k} \), and \( \mathbf{r}(0) = (1,1,1) \), \( \mathbf{r}(1) = (2,e,2) \).

Then
\[
\int_C \mathbf{G}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} + \int_C z \, dx - y^2 \, dz
\]
\[
= f(2,e,2) - f(1,1,1) + \int_0^1 [(1 + u)2u - e^{2u}] \, du (4 \text{ points})
\]
\[
= -e^{-2} + 2e^2 - 4 + \left( \frac{2}{3} u^3 + u^2 - \frac{1}{2} e^{2u} \right)_0^1 (4 \text{ points})
\]
\[
= \frac{3}{2} e^2 - e^{-2} - \frac{11}{6} (2 \text{ points})
\]
5. (15%) Let $C$ be a piecewise-smooth Jordan curve that does not pass through the origin. Evaluate $\int_C \frac{-y^5}{(x^2 + y^2)^3} \, dx + \frac{xy^4}{(x^2 + y^2)^3} \, dy$ for the following two cases, where $C$ is traversed in the counterclockwise direction.

(a) $C$ does not enclose the origin.

(b) $C$ does enclose the origin.

Solution:

(a) Let $\Omega$ be the region enclosed by $C$. Since $\Omega$ does not enclose the origin, the functions

$$P(x, y) = \frac{-y^5}{(x^2 + y^2)^3}$$

and

$$Q(x, y) = \frac{xy^4}{(x^2 + y^2)^3}$$

are well-defined and differentiable in $\Omega$. We have

$$\frac{\partial Q}{\partial x}(x, y) = \frac{\partial P}{\partial y}(x, y) = \frac{y^6 - 5x^2y^4}{(x^2 + y^2)^4}$$

in $\Omega$. Therefore, by Green’s theorem,

$$\oint_C P \, dx + Q \, dy = \iint_\Omega \left( \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \right) \, dxdy = 0.$$

Grading Policy:

Application of Green’s theorem: 2%
Correct calculation of partial derivatives: 2%
Correct answer: 3%

(b) Let $C_r$ be the curve $\theta \mapsto (r \cos \theta, r \sin \theta)$, $\theta \in [0, 2\pi]$, where $r$ is small enough such that $C_r$ lies in the interior of the region bounded by $C$. Let $\Omega$ be the region bounded by $C$ and $C_r$. By Green’s theorem, we have

$$\oint_C P \, dx + Q \, dy - \oint_{C_r} P \, dx + Q \, dy = \iint_\Omega \left( \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \right) \, dxdy,$$

where $P(x, y)$ and $Q(x, y)$ are defined as in (a). Since $\Omega$ does not contain the origin, the right hand side of the above equation is 0. Thus

$$\oint_C P \, dx + Q \, dy = \oint_{C_r} P \, dx + Q \, dy = \int_0^{2\pi} \left( \frac{r^5 \sin^5 \theta}{r^6} (-r \sin \theta) + \frac{r^5 \cos \theta \sin^4 \theta}{r^6} (r \cos \theta) \right) \, d\theta$$

$$= \int_0^{2\pi} \sin^4 \theta d\theta$$

$$= \frac{3\pi}{4}.$$

Grading Policy:

Valid application of Green’s theorem: 3%
Correctly transforming the line integral to the ordinary integral: 2%
Correct answer: 3%
6. (15%) Let $S$ be the triangular region with vertices $(0,0,0)$, $(a,0,0)$, and $(a,a,a)$, $a > 0$, with upward unit normal $n$, and $C$ be the positively oriented boundary of $S$. Let

$$F = (y - z \cos(x^2)) \mathbf{i} + (2x - \sin(z^2)) \mathbf{j} + (3z - \tan(y^2)) \mathbf{k}.$$ 

(a) Find a parametrization of $S$ and find the upward unit normal $n$. (Hint. Consider the projection of $S$ to $xy$-plane.)

(b) Evaluate $\nabla \times F$.

(c) Evaluate $\int_C F \cdot d\mathbf{r}$.

Solution:

(a)  
\begin{enumerate}
\item $S : \{(x, y, y) \mid 0 \leq y \leq x \leq a \}$ (2 points)
\item $n = \frac{1}{\sqrt{2}} (0, -1, 1)$ (2 points)
\end{enumerate}

(b) 

$$\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_x & F_y & F_z
\end{vmatrix}$$

$$= \left[ 2z \cos z^2 - 2y \sec^2 y^2 \right] \mathbf{i} - \cos x^2 \mathbf{j} + \mathbf{k}$$

(c) $\int_C F \cdot d\mathbf{r} = \iint_S \nabla \times F \cdot \mathbf{n} \, d\sigma = \int_0^a \int_0^x \frac{1}{\sqrt{2}} (1 + \cos x^2) \cdot |\mathbf{r}_x \times \mathbf{r}_y| \, dy \, dx$ (4 points)

$$= \int_0^a x(1 + \cos x^2) \, dx = \frac{x^2 + \sin x^2}{2} \bigg|_0^a = \frac{1}{2} (a^2 + \sin a^2)$$ (3 points)
7. (15%) Let \( S_1 \) be the surface \( \{(x, y, z) | z = x^2 + y^2, \ z \leq y\} \), \( S_2 \) be the surface \( \{(x, y, z) | z = y, \ x^2 + y^2 \leq z\} \), and \( \mathbf{V}(x, y, z) = -yi + xj + zk \).

(a) Compute directly the downward flux of \( \mathbf{V} \) across \( S_1 \).

(b) Use the divergence theorem to compute the upward flux of \( \mathbf{V} \) across \( S_2 \).

**Solution:**

7. (a)

\[
S_1 : f_1(x, y) = (x, y, x^2 + y^2), \ x^2 + y^2 \leq y
\]

\[
\frac{\partial f_1}{\partial x} = (1, 0, 2x)
\]

\[
\frac{\partial f_1}{\partial y} = (0, 1, 2y)
\]

\[
\frac{\partial f_1}{\partial x} \times \frac{\partial f_1}{\partial y} = (-2x, -2y, 1)
\]

\[
\mathbf{d} \text{ area} = (2x, 2y, -1) \ dx \ dy \quad \text{(since the direction is downward)} \quad (1 \ pt)
\]

\[
\mathbf{V} = (-y, x, z)
\]

Let \( x = r \cos \theta, y = r \sin \theta \). Then \( r^2 \leq r \sin \theta \Rightarrow 0 \leq r \leq \sin \theta, 0 \leq \theta \leq \pi \).

\[
\iint_{S_1} \mathbf{V} \cdot \mathbf{d} \text{ area} = \int_{\theta=0}^{\pi} \int_{r=0}^{\sin \theta} -2xy + 2xy - (x^2 + y^2) \ dx \ dy \quad (1 \ pt)
\]

\[
= \int_{\theta=0}^{\pi} \int_{r=0}^{\sin \theta} -r^2 \ r \ dr \ d\theta \quad (3 \ pts)
\]

\[
= \left[ -\frac{r^3}{4} \right]_{r=0}^{r=\sin \theta} \ d\theta
\]

\[
= \left[ -\frac{3}{4} \sin^4 \theta \right] \ d\theta
\]

\[
= \left. \frac{1}{12} \left( \frac{3}{4} \pi \right) \right| = \frac{\pi}{32} \quad (2 \ pts)
\]

(b)

Let \( x = r \cos \theta, y = r \sin \theta \).

\[
\Omega : x^2 + y^2 \leq z \leq y \Rightarrow \begin{cases} r^2 \leq z \leq y \\ r^2 \leq r \sin \theta \end{cases} \Rightarrow \begin{cases} r^2 \leq z \leq y \\ 0 \leq r \leq \sin \theta \end{cases} \quad 0 \leq \theta \leq \pi
\]

By divergent theorem,

\[
\iiint_{\Omega} \nabla \cdot \mathbf{V} \ d \text{ volume} = \iint_{\partial \Omega = S_1 + S_2} \mathbf{V} \cdot \mathbf{d} \text{ area} = \iint_{S_1} \mathbf{V} \cdot \mathbf{d} \text{ area} + \iint_{S_2} \mathbf{V} \cdot \mathbf{d} \text{ area} \quad (1 \ pt).
\]

\[
\iiint_{\Omega} \nabla \cdot \mathbf{V} \ d \text{ volume} = \int_{\theta=0}^{\pi} \int_{r=0}^{\sin \theta} \int_{z=r^2}^{r \sin \theta} 1 \ r \ dz \ dr \ d\theta \quad (3 \ pts)
\]

\[
= \int_{\theta=0}^{\pi} \int_{r=0}^{\sin \theta} (r \sin \theta - r^2) \ r \ dr \ d\theta
\]

\[
= \int_{\theta=0}^{\pi} \frac{r^3}{3} \sin \theta \ |_{0}^{\sin \theta} - \frac{r^4}{4} \ |_{0}^{\sin \theta} \ d\theta
\]

\[
= \frac{r^3}{3} \left[ \sin^4 \theta \right]_{0}^{\sin \theta} - \frac{r^4}{4} \left[ \sin^4 \theta \right]_{0}^{\sin \theta} \ d\theta
\]

\[
= \left[ \frac{1}{12} \left( \frac{3}{4} \pi \right) \right] = \frac{\pi}{32} \quad (2 \ pts)
\]

\[
\Rightarrow \iint_{S_2} \mathbf{V} \cdot \mathbf{d} \text{ area} = \frac{\pi}{32} - \left( \frac{3\pi}{32} \right) = \frac{\pi}{8} \quad (2 \ pts)
\]