1. (16 points) Prove that the series \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1) 2^n} \) converges absolutely, and find its sum.

**Solution:**

Part I (6pts) Prove the series converges absolutely

Let \( a_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1) 2^n} \), then the series convergent absolutely if \( \sum_{n=2}^{\infty} |a_n| \) converges.

Use ratio test:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{1/(n+1)n2^{n+1}}{1/(n)2^n} \right| = \frac{1}{2} < 1
\]

Use comparison test:

Compared \( a_n \) to \( b_n = \frac{1}{n^2} \) or \( b_n = \frac{1}{2^n} \)

Show that \( a_n < b_n \) everywhere and prove that \( \sum_{n=2}^{\infty} |b_n| \) converges.

Part II (10pts) Find its sum

\[
f(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 ... = \sum_{n=0}^{\infty} (-1)^n x^n
\]

\[
g(x) = \int \frac{1}{1+x} \, dx = \ln(1+x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 ... + c1 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n + c2
\]

For \( x = 0 \rightarrow c1 = c2 = 0 \)

\[
h(x) = \int \ln(1+x) \, dx = \frac{1}{1 \times 2} x^2 - \frac{1}{2 \times 3} x^3 + \frac{1}{3 \times 4} x^4 ... + c3 = \sum_{n=2}^{\infty} (-1)^n \frac{1}{n(n-1)} x^n + c4
\]

\[
(1+x) \ln(1+x) - (1+x) = \sum_{n=2}^{\infty} (-1)^n \frac{1}{n(n-1)} x^n + c4
\]

For \( x = 0 \rightarrow c4 = -1 \)

Let \( x = \frac{1}{2} \)

\[
\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(n-1)} x^n = \frac{3}{2} \ln \frac{3}{2} - \frac{1}{2}
\]

2. (a) (8 points) The Binomial Theorem implies that

\[
(1 - x^{-\frac{1}{2}}) = 1 + \sum_{n=1}^{\infty} \frac{(2n)!}{k^n(n!)^2} x^n
\]

for some constant \( k \). Find \( k \), and find the interval of convergence of the power series.

(b) (8 points) Estimate the error if one uses \( x = -\frac{1}{4} \), and the first five non-zero terms in (a) to approximate \( \frac{1}{\sqrt{5}} \).

**Solution:**
3. (16 points) Consider the curve 
\[ r(t) = t^2 \mathbf{i} + (\sin t - t \cos t) \mathbf{j} + (\cos t + t \sin t) \mathbf{k}, \quad t \geq 0 \]
Find \( T(t), N(t), B(t) \), the curvature \( \kappa(t) \) and the torsion \( \tau(t) \). 

**Solution:**
\[
r(t) = (t^2, \sin t - t \cos t, \cos t + t \sin t)
\]
\[
r'(t) = v(t) = (2t, \cos t + tsint - cost, -sint + tcost + sint) = (2t, tsint, tcost)
\]
\[
r''(t) = a(t) = (2, sint + tcost, cost - tsint)
\]
\[
r'''(t) = a'(t) = (0, 2cost - tsint, -2sint - tcost)
\]
\[
|v'(t)| = \sqrt{4t^2 + t^2 \sin^2 t + t^2 \cos^2 t} = \sqrt{5}t
\]
\[
T(t) = \frac{r'(t)}{|v'(t)|} = \frac{1}{\sqrt{5}}(2, sint, cost) = \frac{\sqrt{5}}{5}(2, sint, cost)
\]
To get full points (3%) please answer both question correctly.
\[
T''(t) = \frac{1}{\sqrt{5}}(0, cost, -sint), \quad |T''(t)| = \frac{1}{\sqrt{5}} \sqrt{\cos^2 t + \sin^2 t} = \frac{1}{\sqrt{5}}
\]
\[
N(t) = \frac{T''(t)}{|T''(t)|} = B(t) \cdot N(t) = (0, cost, -sint)
\]
To get full points (3%) please answer both question correctly.
\[
\kappa(t) = \frac{|v(t) \times a(t)|}{|v'(t)|^3} = \frac{1}{\sqrt{5}} \sqrt{\frac{1}{5}t^3} = \frac{1}{5t}
\]
To get full points (3%) please answer both question correctly.
\[
v(t) \times a(t) = (-t^2, 2t^2 sint, 2t^2 cost)
\]
\[
|v(t) \times a(t)| = \sqrt{(-t^2)^2 + (2t^2 sint)^2 + (2t^2 cost)^2} = \sqrt{t^4 + 4t^4} = \sqrt{5}t^2
\]
If your calculation is wrong, you will receive some partial credit.
\[
B(t) = \frac{v(t) \times a(t)}{|v(t) \times a(t)|} = \frac{1}{\sqrt{5}}(-1, 2sint, 2cost) = \frac{\sqrt{5}}{5}(-1, 2sint, 2cost)
\]
To get full points (3%) please answer both question correctly.
\[
\tau(t) = \frac{|v(t) \times a(t) \cdot a'(t)|}{|v(t) \times a(t)|^2} = \frac{2t^2 sint(2cost - tsint) + 2t^2 cost(-2sint - tcost)}{5t^4}
\]
\[
= \frac{(4t^2 sint \cdot cost - 2t^4 \sin^2 t) + (-4t^2 sint \cdot cost - 2t^2 \cos^2 t)}{5t^4}
\]
\[
= \frac{-2t^3}{5t^4}
\]
I will also grant partial credit for partial solutions and solutions with minor flaws. I will give no credit for wildly incorrect answers which are obviously only there in the hopes of getting partial credit.
4. (16 points) Let \( F(x, y, z) = x^2 + 2z + \int_y^2 \sqrt{(t^2 + 7)y^2} dt \). Find the tangent plane of the surface \( F(x, y, z) = 2 \) at the point \((2, -1, -1)\).

Solution:
\[
F_1 = \frac{\partial F}{\partial x} = 2x \quad \text{(1 point)}
\]
\[
F_2 = \frac{\partial F}{\partial y} = 2y \cdot \frac{1}{3} \int_y^2 \sqrt{t^2 + 7} dt - y^2 \sqrt{y^2 + 7} \quad \text{(5 points)}
\]
\[
F_3 = \frac{\partial F}{\partial z} = 2 + \frac{2}{3} y^2 \sqrt{z^2 + 7} \quad \text{(4 points)}
\]
\[
F_1(2, -1, -1) = 4 \quad \text{(1 points)}
\]
\[
F_2(2, -1, -1) = -2 \quad \text{(2 points)}
\]
\[
F_3(2, -1, -1) = 4 \quad \text{(1 points)}
\]
\[
4(x - 2) - 2(y + 1) + 4(z + 1) = 0 \quad \text{(2 points)}
\]

5. Suppose that \( z = f(x, y) \) has continuous second order partial derivatives, and \( x = s^2 - t^2, y = 2st \). Define \( F(s, t) = f(s^2 - t^2, 2st) \).

(a) (6 points) Express \( F_s, F_t \) in terms of \( f_x, f_y, s, \) and \( t \).

\[
F_s(s, t) = 2sf_x(s^2 - t^2, 2st) + 2tf_y(s^2 - t^2, 2st) \quad \text{..........................(3pts)}
\]
\[
F_t(s, t) = -2tf_x(s^2 - t^2, 2st) + 2sf_y(s^2 - t^2, 2st) \quad \text{..........................(3pts)}
\]

(b) (10 points) Show that \( F_{ss} + F_{tt} = h(s, t)(f_{xx} + f_{yy}) \) for some function \( h(s, t) \). Find \( h(s, t) \) explicitly.

Solution:
5.(a)
\[
F_s(s, t) = 2sf_x(s^2 - t^2, 2st) + 2tf_y(s^2 - t^2, 2st) ..........................(3pts)
\]
\[
F_t(s, t) = -2tf_x(s^2 - t^2, 2st) + 2sf_y(s^2 - t^2, 2st) ..........................(3pts)
\]
\[
F_{ss}(s, t) + F_{tt}(s, t) = 4(s^2 + t^2)f_{xx} + 4(s^2 + t^2)f_{yy} ..........................(2pts)
\]
\[
= 4(s^2 + t^2)(f_{xx} + f_{yy}) \Rightarrow h(s, t) = 4(s^2 + t^2)........(2pts)
\]

6. Let \( f(x, y, z) = xy^2 + xz^2 - y \).

(a) (10 points) Find all critical points of \( f(x, y, z) \) and classify them.

(b) (10 points) Find the maximum and minimum of \( f \) on the region \( x^2 + y^2 + z^2 \leq 1 \).

Solution:
(a)
\[
f_1 = 2xy + z^2 \quad \text{(1 point)} \Rightarrow \text{critical points at } f_1 = f_2 = f_3 = 0
\]
\[
f_2 = x^2 - 1 \quad \text{(1 point)} \Rightarrow (\pm 1, 0, 0) \quad (2 \text{ points})
\]
\[
f_3 = 2xz \quad \text{(1 point)}
\]
\[
\text{Hessian} \begin{pmatrix} 2y & 2x & 2z \\ 2x & 0 & 0 \\ 2z & 0 & 2x \end{pmatrix} \quad \text{(1 point)} \Rightarrow \begin{pmatrix} 0 & \pm 2 & 0 \\ \pm 2 & 0 & 0 \\ 0 & 0 & \pm 2 \end{pmatrix} \quad (2 \text{ points})
\]
\[
\text{def } H \neq 0 \Rightarrow \text{neither positive definite nor negative definite}
\]
\[
\Rightarrow \text{both saddle points} \quad (2 \text{ points})
\]
(b) \[ x^2 + y^2 + z^2 - 1 = g(x, y, z) \leq 0 \]

Solve the system \[
\begin{cases}
\nabla f = \lambda \nabla g \quad (1 \text{ point}) \\
g = 0
\end{cases}
\]

\[
\begin{align*}
2xz + z^2 &= 2\lambda x \\
x^2 - 1 &= 2\lambda y \\
2xz &= 2\lambda z \\
x^2 + y^2 + z^2 &= 1
\end{align*}
\]

Case 1: \( z = 0 \), then \( 2xy = 2\lambda x \). If \( x = 0 \), \( y = \pm 1 \) \( \Rightarrow (0, \pm 1, 0) \Rightarrow f(0, \pm 1, 0) = \mp 1 \)
If \( x \neq 0 \), \( y = \lambda \Rightarrow x^2 = 2y^2 + 1 = 1 - y^2 \Rightarrow (\pm 1, 0, 0) \Rightarrow f(\pm 1, 0, 0) = 0 \) (1 point)

Case 2: \( z \neq 0 \), \( x = \lambda \Rightarrow x^2 = 2xy + 1 = (2x^2 - z^2) + 1 \Rightarrow (0, 0, \pm 1) \Rightarrow f(0, 0, \pm 1) = 0 \) (1 point)
\max = 1 \text{ at } (0, -1, 0) \quad (2 \text{ points}), \quad \min = -1 \text{ at } (0, 1, 0) \quad (2 \text{ points})