Solutions of Homework #1

1. By the assumption, \( S = \sigma(A) = \sigma\)-algebra generated by \( A \).
Let \( F = \{ \mathcal{E} \subset A | \mathcal{E} \text{ has countable elements} \} \). We want to show that \( S = \bigcup_{\mathcal{E} \in F} \sigma(\mathcal{E}) \).

1° claim: \( \bigcup_{\mathcal{E} \in F} \sigma(\mathcal{E}) \) is also a \( \sigma \)-algebra containing \( A \).

pf: \( \bigcup_{\mathcal{E} \in F} \sigma(\mathcal{E}) \supset A \) is obviously.

(i) \( \forall \{E_n\}_{n=1}^{\infty} \subset \bigcup_{\mathcal{E} \in F} \sigma(\mathcal{E}) \), there exists \( \mathcal{E}_n \in F \) such that \( E_n \in \mathcal{E}_n, \forall n \). Since \( \mathcal{E}_n \) has countable elements, \( \bigcup_{n=1}^{\infty} \mathcal{E}_n \) has countable elements.
Let \( \bigcup_{n=1}^{\infty} \mathcal{E}_n = \mathcal{E}_0 \in F \), we have \( E_n \in \mathcal{E}_0, \forall n \).
Since \( \sigma(\mathcal{E}_0) \) is a \( \sigma \)-algebra, then \( \bigcup_{n=1}^{\infty} E_n \in \bigcup_{\mathcal{E} \in F} \sigma(\mathcal{E}) \).

(ii) Given \( E_1, E_2 \in \bigcup_{\mathcal{E} \in F} \sigma(\mathcal{E}) \). Similarly to (i), there exists \( \mathcal{E}_1 \) such that \( E_1, E_2 \in \sigma(\mathcal{E}_1) \). Therefore, \( E_1 \cap E_2 \in \sigma(\mathcal{E}_1) \subset \bigcup_{\mathcal{E} \in F} \sigma(\mathcal{E}) \).
Moreover, \( \emptyset, X \in \bigcup_{\mathcal{E} \in F} \sigma(\mathcal{E}) \). Therefore by (i) and (ii), \( \bigcup_{\mathcal{E} \in F} \sigma(\mathcal{E}) \) is a \( \sigma \)-algebra containing \( A \).

Since \( S \) is the \( \sigma \)-algebra generated by \( A \), by the claim, \( S \subset \bigcup_{\mathcal{E} \in F} \sigma(\mathcal{E}) \).

2° On the other hand, for each \( \mathcal{E} \in F, \mathcal{E} \subset A \Rightarrow \sigma(\mathcal{E}) \subset \sigma(A) = S \), \( \forall \mathcal{E} \in F \).
\( \Rightarrow \bigcup_{\mathcal{E} \in F} \sigma(\mathcal{E}) \subset S \).

By 1° and 2°, \( \bigcup_{\mathcal{E} \in F} \sigma(\mathcal{E}) = S \)

2. Since \( A \subset 2^X \) is an algebra, \( \therefore X \in A \).
Let \( \mathcal{M}(\mu^*) = \) the collection of \( \mu^* \)-measurable sets.

(i) Given \( E \subset X \), we always can find \( A_n \subset A \) such that \( E \subset \bigcup_{n=1}^{\infty} A_n. (\therefore X \in A) \)
Define
\[
\mu^*(E) = \inf \{ \sum_{n=1}^{\infty} \mu(A_n) | E \subset \bigcup_{n=1}^{\infty} A_n, A_n \in A \}.
\]
Therefore \( \forall \varepsilon > 0, \exists \{A_n\}_{n=1}^{\infty} \subset A \) with \( E \subset \bigcup_{n=1}^{\infty} A_n \) such that
\[
\sum_{n=1}^{\infty} \mu(A_n) < \mu^*(E) + \varepsilon
\]
Let \( A = \bigcup_{n=1}^{\infty} A_n. \Rightarrow A \in A_\sigma. \)
\[
\mu^*(A) = \inf \{ \sum_{n=1}^{\infty} \mu(B_n) | A \subset \bigcup_{n=1}^{\infty} B_n, B_n \in A \}
\]
\[
\leq \sum_{n=1}^{\infty} \mu(A_n)
\]
\[
\leq \mu^*(E) + \varepsilon
\]
(ii) $\mu^*(E) < \infty$.

$(\Rightarrow)$ Suppose that $E$ is $\mu^*$-measurable.

Given $F \subset X$,

$$\mu^*(F) = \mu^*(F \cap E) + \mu^*(F \setminus E) \quad (1)$$

By (i), we have $\forall n \in \mathbb{N}, \exists A_n \in \mathcal{A}_\sigma$ such that

$$\mu^*(A_n) \leq \mu^*(E) + \frac{1}{n}, \forall n.$$  

Let $B = \bigcap_{n=1}^{\infty} A_n \in \mathcal{A}_{\delta}$ $\Rightarrow E \subset B$. Since $B \subset A_n, \forall n$, we have

$$\mu^*(B) \leq \mu^*(A_n) \leq \mu^*(E) + \frac{1}{n}, \forall n.$$  

$\Rightarrow \mu^*(B) \leq \mu^*(E)$.

By $E \subset B$, $\mu^*(B) = \mu^*(E)$. By (1), we have

$$\mu^*(E) = \mu^*(B) = \mu^*(B \cap E) + \mu^*(B \setminus E) = \mu^*(E) + \mu^*(B \setminus E).$$

Since $\mu^*(E) < \infty$, we have $\mu^*(B \setminus E) = 0$

$(\Leftarrow)$ Suppose that there exists $B \in (A)_{\delta \sigma}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0 \Rightarrow B \setminus E$ is $\mu^*$-measurable.

Since $E = B \setminus (B \setminus E)$ and $B, B \setminus E$ are $\mu^*$-measurable, we have $E$ is also $\mu^*$-measurable. $(\because \mathcal{M}(\mu^*)$ is a $\sigma$-algebra.)

(iii) Since $\mu$ is $\sigma$-finite, there exists $\{X_n\}_{n=1}^{\infty} \subset \mathcal{A}$ such that

$$X = \bigcup_{n=1}^{\infty} X_n \text{ and } \mu(X_n) < \infty$$

$(\Leftarrow)$ This proof is the same as $\Leftarrow$ of (ii).

$(\Rightarrow)$ Suppose that $E$ is $\mu^*$-measurable. Then $E = \bigcup_{n=1}^{\infty} (E \cap X_n)$.

Since $\mathcal{M}(\mu^*)$ is a $\sigma$-algebra and $X_n \in \mathcal{A} \subset \mathcal{M}(\mu^*)$,

Let $E_n = E \cap X_n \in \mathcal{M}(\mu^*), \forall n$.

Moreover, we have $\mu^*(E_n) \leq \mu^*(X_n) < \infty, \forall n$.

By (i), $\forall k \in \mathbb{N}, \exists B_{n,k} \in A_\sigma$ with $B_{n,k} \supset E_n$ and

$$\mu^*(B_{n,k}) \leq \mu^*(E_n) + \frac{1}{k2^n}.$$  

Since $\mu^*(E_n) < \infty$ and $E_n \in M(\mu^*)$, we have

$$\mu^*(B_{n,k} \setminus E_n) = \mu^*(B_{n,k}) - \mu^*(E_n) \leq \frac{1}{k2^n}.$$  

Therefore $\sum_{n=1}^{\infty} \mu^*(B_{n,k} \setminus E_n) \leq \frac{1}{k}$.

Let $B^{(k)} = \bigcup_{n=1}^{\infty} B_{n,k} \in A_\sigma \Rightarrow E \subset B^{(k)}$.

$$\mu^*(B^{(k)} \setminus E) \leq \mu^*(\bigcup_{n=1}^{\infty} (B_{n,k} \setminus E_n)) \leq \sum_{n=1}^{\infty} \mu^*(B_{n,k} \setminus E_n) \leq \frac{1}{k}.$$
Let \( B = \bigcap_{k=1}^{\infty} B^{(k)} \). Then
\[
\mu^*(B \setminus E) \leq \mu^*(B^{(k)} \setminus E) \leq \frac{1}{k}, \forall k.
\]

Therefore \( \mu^*(B \setminus E) = 0 \) and \( E \subset B, B \in \mathcal{A}_{\sigma} \).
Hence \( B \) is what we want.

3. \((\Rightarrow)\) Suppose that \( E \) is \( \mu^* \)-measurable. Then by the definition of measurable sets, we have
\[
\mu^*(X) = \mu^*(X \cap E) + \mu^*(X \setminus E)
\]

Therefore, by \( \mu(X) < \infty \), we obtain
\[
\mu^*(E) = \mu(X) - \mu^*(E^c) = \mu_e(E)
\]

\((\Leftarrow)\) Suppose that \( \mu^*(E) = \mu_e(E) \). We have
\[
\mu^*(E) = \mu_e(E) = \mu(X) - \mu^*(E^c).
\] (2)

We give two proofs.

proof(a): For any \( F \subset X \), \( \forall n \in \mathbb{N} \), \( \exists A_n \in \mathcal{A}_\sigma \) with \( F \subset A_n \) such that
\[
\mu^*(A_n) \leq \mu^*(F) + \frac{1}{n}.
\]
Thus, \( \mu^*(\cap A_n) \leq \mu^*(A_n) \leq \mu^*(F) + \frac{1}{n}, \forall n. \)
Therefore, we have \( \mu^*(\cap A_n) = \mu^*(F). \)

Let \( \mathcal{M}(\mu^*) = \) the collection of \( \mu^* \)-measurable sets. Since \( \mathcal{A} \subset \mathcal{M}(\mu^*) \) and \( \mathcal{M}(\mu^*) \) is a \( \sigma \)-algebra, we know that \( \mathcal{A}_\sigma \subset \mathcal{M}(\mu^*). \)
So, \( \cap A_n \in \mathcal{M}(\mu^*). \) So far, we have the following conclusion:
For any \( F \subset X \), there exists a \( \mu^* \)-measurable set \( B \) such that
\( B \supseteq F \) and \( \mu^*(F) = \mu^*(B). \)

Thus, we pick two measurable sets \( B_1, B_2 \) such that
\[
B_1 \supseteq E, B_2 \supseteq E^c \quad \text{and} \quad \mu^*(B_1) = \mu^*(E), \mu^*(B_2) = \mu^*(E^c).
\]

By (2), we have \( \mu(B_1) + \mu(B_2) = \mu(X) \). Since \( B_1 \) and \( B_2 \) are \( \mu^* \)-measurable, we have
\[
\mu(X) = \mu(B_1) + \mu(B_2) = \mu(B_1 \cap B_2) + \mu(B_1 \setminus B_2) + \mu(B_2) = \mu(B_1 \cap B_2) + \mu(B_1 \cup B_2) = \mu(B_1 \cap B_2) + \mu(X).
\]

Since \( \mu(X) < \infty \), we have \( \mu(B_1 \cap B_2) = 0 \). Thus, \( \mu(B_1 \setminus B_2) = 0 \). And we have \( B_1 \cap B_2 \) is \( \mu^* \)-measurable.
By \( E = B_1 \setminus (B_1 \cap B_2) \), we have \( E \) is \( \mu^* \)-measurable.

proof(b): For any \( F \subset X \), \( \forall \varepsilon > 0 \), \( \exists A_\varepsilon \in \mathcal{A}_\sigma \) with \( F \subset A_\varepsilon \) such that
\[
\mu^*(A_\varepsilon) \leq \mu^*(F) + \varepsilon.
\]

Let \( \mathcal{M}(\mu^*) = \) the collection of \( \mu^* \)-measurable sets. Since \( \mathcal{A} \subset \mathcal{M}(\mu^*) \) and \( \mathcal{M}(\mu^*) \) is a \( \sigma \)-algebra, we know that \( \mathcal{A}_\sigma \subset \mathcal{M}(\mu^*). \)
So, \( \mu(X) = \mu^*(A_\varepsilon) + \mu^*(A_\varepsilon^c). \)
By countably subadditivity of $\mu^*$, we obtain the following inequality
\[
\mu(X) = \mu^*(A_\varepsilon) + \mu^*(A_\varepsilon^c) \\
\leq \mu^*(A_\varepsilon \cap E) + \mu^*(A_\varepsilon \cap E^c) + \mu^*(A_\varepsilon^c \cap E) + \mu^*(A_\varepsilon^c \cap E^c) \\
= \mu^*(E) + \mu^*(E^c) \quad \text{(because $A_\varepsilon \in M(\mu^*)$.)} \\
= \mu(X) \quad \text{(by (2))}
\]
Therefore, we in fact have
\[
\mu^*(A_\varepsilon) = \mu^*(A_\varepsilon \cap E) + \mu^*(A_\varepsilon \cap E^c) \\
\mu^*(A_\varepsilon^c) = \mu^*(A_\varepsilon^c \cap E) + \mu^*(A_\varepsilon^c \cap E^c)
\]
Since $F \cap A_\varepsilon$, we have
\[
\mu^*(F \cap E) + \mu^*(F \cap E^c) \leq \mu^*(A_\varepsilon \cap E) + \mu^*(A_\varepsilon \cap E^c) \\
= \mu^*(A_\varepsilon) \leq \mu^*(F) + \varepsilon
\]
Since $\varepsilon$ is arbitrary, $\mu^*(F \cap E) + \mu^*(F \cap E^c) \leq \mu^*(F)$. The opposite inequality is obvious, therefore $E$ is $\mu^*$-measurable.

4. (i) By the following two theorem (one is proved in the class and the other is showed in Folland, Real Analysis):

**Theorem 1.** (Carathéodory extension) Let $\nu$ be a countably additive on a ring $R$ and $\nu : R \to [0, \infty]$. There exists a measure on a $\sigma$-algebra, that coincides with $\nu$ on $R$. (Indeed, )

**Theorem 2.** (Folland, Real Analysis, Theorem 1.14)

We know we can extend $\mu$ from a ring or an algebra to a $\sigma$-algebra if $\mu$ is countably additive. But notice that $E$ is not a ring! For instance, we may define $A_1 = (-2, -1] \cup (1, 2]$ and $A_2 = (-4, -3] \cup (3, 4]$. Thus it’s easy to see $A_1 \cup A_2$ doesn’t belong to $\mathcal{E}$.

Therefore we need to find a way to prove this problem. Here are 2 methods to prove it, but the ideas are essentially the same. Since if we write down them all, the proof becomes too long, we only show the sketches.

1) Define $\mathcal{R} = \{\emptyset\} \cup \{E | E = \bigcup_{n=1}^{m} A_{a_n, b_n}, \text{ for some } m \in \mathbb{N} \text{ and } A_{a_n, b_n} \in \mathcal{E} \text{ are mutually disjoint}\}.$

Hence we can define $\tilde{\mu}$ on $\mathcal{R}$ by
\[
\tilde{\mu}\left(\bigcup_{n=1}^{m} A_{a_n, b_n}\right) = \sum_{n=1}^{m} \mu(A_{a_n, b_n}) , \text{and } \tilde{\mu}(\emptyset) = 0
\]
Check $\tilde{\mu}$ is countable additive on $\mathcal{R}$ and $\tilde{\mu} = \mu$ on $\mathcal{E}$.

By Theorem 1, there exists a measure on a $\sigma$-algebra, that coincides with $\mu$ on $\mathcal{R}$ and therefore on $\mathcal{E}$.
2) Show that $\mathcal{E}' = \mathcal{E} \cup \emptyset$ is a semi-ring.

That is $\mathcal{E}'$ satisfies the following properties:

a. $\emptyset \in \mathcal{E}'$

b. $A, B \in \mathcal{E}' \Rightarrow A \cap B \in \mathcal{E}'$

c. $A, B \in \mathcal{E}' \Rightarrow \exists n \geq 0, \exists A_i \in \mathcal{E}'$ are disjoint s.t. $A \setminus B = \bigcup_{i=1}^{n} A_i$

Use the following theorem, then we can a extension of $\mu$ to a $\sigma$-algebra:

**Theorem 3.** Let $\mathcal{S}$ be a semi-ring on $X$ and $\mu : \mathcal{S} \rightarrow [0, \infty]$ be a measure on $\mathcal{S}$. There exists a measure $\bar{\mu} : \sigma(\mathcal{S}) \rightarrow [0, \infty]$ such that $\bar{\mu} = \mu$ on $\mathcal{S}$.

(ii) $[1, 2]$ is NOT a $\mu^*$-measurable!

By definition of $\mu^*$:

$$\mu^*(E) = \inf \left\{ \sum_n \mu(A_n) | E \subset \bigcup_n A_n, A_n \in \mathcal{A} \right\},$$

therefore $\mu^*([1, 2]) = \mu^*([-2, -1]) = 1$.

Suppose to the contrary that $[1, 2]$ is $\mu^*$-measurable, then

$$1 = \mu^*([-2, -1] \cup [1, 2]) = \mu^*([1, 2]) + \mu^*([-2, -1]) = 1 + 1 = 2$$

Therefore we get a contradiction! Hence $[1, 2]$ is not $\mu^*$-measurable.