Unique continuation for the elasticity system and a counterexample for second order elliptic systems

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Abstract

In this paper we study the global unique continuation property for the elasticity system and the general second order elliptic system in two dimensions. For the isotropic and the anisotropic systems with measurable coefficients, under certain conditions on coefficients, we show that the global unique continuation property holds. On the other hand, for the anisotropic system, if the coefficients are Lipschitz, we can prove that the global unique continuation is satisfied for a more general class of media. In addition to the positive results, we also present counterexamples to unique continuation and strong unique continuation for general second elliptic systems.

1 Introduction

In this work, we study the unique continuation property for the elasticity system and the general second order elliptic system in two dimensions. We begin with the elasticity system. Let $u = (u_1, u_2)^T$ be a vector-valued function satisfy

$$\partial_j(a_{ijkl}(x)\partial_k u_l) = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{1.1}$$

where $a_{ijkl}(x)$ is a rank four tensor satisfying the symmetry properties:

$$a_{ijkl} = a_{klij} = a_{jikl}. \tag{1.2}$$

Throughout, the Latin indices range from 1 to 2. Also, the summation convention is imposed. For isotropic media, we have that $a_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where λ and μ are called Lamé coefficients. In this case, (1.1) is written as

$$\nabla \cdot (\mu(\nabla u + (\nabla u)^T)) + \nabla(\lambda\nabla \cdot u) = 0 \quad \text{in} \quad \mathbb{R}^2. \tag{1.3}$$

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We say that u, a solution of (1.3), satisfies the *global unique continuation property* if whenever u that vanishes in the lower half plane, it vanishes identically in $\mathbb{R}^2$. Recall that any solution u of the partial differential equation defined in an open connected set Ω is said to satisfy the unique continuation property if whenever u vanishes in a non-empty open subset of Ω, it is zero in Ω. On the other hand, u satisfies the strong unique continuation property if whenever u vanishes of infinite order at any point of Ω, it vanishes in Ω.

For the isotropic system (1.3) with nice Lamé coefficients, there are a lot of results on the unique continuation property and the strong unique continuation property (for dimension $n \geq 2$). We will not review the detailed development here. To motivate our study, we only mention the recent result in [LNUW11], where the strong unique continuation property was proved for $\mu \in W^{1,\infty}$ and $\lambda \in L^\infty$, which is the best known assumption on the coefficients by far. For the scalar second order elliptic equation in non-divergence or divergence form

$$A\nabla^2 u = 0 \quad \text{in} \quad \mathbb{R}^2 \tag{1.4}$$

or

$$\nabla \cdot (A(x)\nabla u) = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{1.5}$$

the strong unique continuation property is satisfied for $A \in L^\infty$ (see, for example, [Al92], [Al12], [AE08], [BN54], [Sc98]). The proof is based on the intimate connection between (1.4) or (1.5) and quasiregular mappings. Therefore, it is a natural question to ask whether the unique continuation or the strong unique continuation hold for (1.3) or even for (1.1) when all coefficients are only measurable.

When μ of (1.3) is Lipschitz, it is known that (1.3) is weakly coupled. Hence, the usual Carleman method will lead us to the unique continuation properties. However, if μ is only measurable, (1.3) is strongly coupled. To the best of our knowledge, the Carleman method has never been successfully applied to strongly coupled systems. The general elasticity system (1.1) is always strongly coupled, regardless of the regularity of coefficients.

In this work we would like to show that solutions u of (1.1) satisfy the global unique continuation property under some restrictions on the measurable coefficients a_{ijkl}. Our approach to prove this result relies on the the connection between (1.1) and the Beltrami system with matrix-valued coefficients (see (2.16)). When this matrix-valued coefficient is sufficiently small (which is satisfied when the coefficients do not deviate too much from a set of constant coefficients), we can follow the arguments in [IVV02] and use the L^p-norm of the Beurling-Ahlfors transform to conclude the result. When the coefficients are only measurable, the set of constant coefficients is rather restricted, see the conditions in Theorem 2.1 and 2.3. If some coefficients of the general system (1.1) are Lipschitz, the global unique continuation is true for coefficients near a larger set of constant coefficients, see Theorem 3.1.

In addition to the positive results mentioned above, we also present a counterexample to unique continuation for a second order elliptic system (in the sense of (4.26)) with measurable coefficients based on the example derived in [IVV02]. The main idea in the construction of the counterexample is to convert the second order elliptic system to a first order elliptic system and match coefficients of the first order system obtained from the example in [IVV02]. Based on the example given in [CP11] (also see related article [Ro09]), using the same argument, we can also construct a counterexample to strong unique continuation for second order elliptic systems with continuous coefficients satisfying the Lagendre-Hadamard condition (2.21) or even the strong convexity condition:

$$a_{ijkl}(x)\xi_k^l\xi_j^i \geq c|\xi|^2 \tag{1.6}$$

for any 2×2 matrix $\xi = (\xi_k^l)$, where c is a positive constant. Note that (1.6) implies the Legendre-Hadamard condition.

We would like to remark that the nontrivial solution u of the counterexample to unique continuation described above vanishes in the lower half plane. It was shown in [MS03] that there exist nontrivial $W^{1,2}(\mathbb{R}^2)$ solution u or Lipschitz solution u whose supports are compact solving second order elliptic system with measurable coefficients satisfying the Legendre-Hadamard condition. This is another counterexample to unique continuation for second order elliptic systems with measurable coefficients. We want to point out that the examples in [MS03] do not exist in second order elliptic systems satisfying the strong convexity condition (1.6). This can be easily seen by the integration by parts. Nonetheless, the strong convexity condition (1.6) does not rule out the existence of the counterexample to unique continuation we constructed in this paper since this nontrivial solution does not necessarily have compact support.

The paper is organized as follows. In Section 2, we prove the global unique continuation property for the Lamé and general anisotropic systems when the measurable elastic coefficients are close to some constant values. In Section 3, we expand the set of constant values when the elastic coefficients are Lipschitz. Finally, in Section 4, we construct counterexamples to unique continuation and strong unique continuation for general second order elliptic systems with measurable coefficients and continuous coefficients, respectively.

2 Elasticity system with measurable coefficients

It is instructive to begin with the isotropic system, i.e., Lamé system (1.3). Assume that $\lambda, \mu \in L^\infty$ satisfy the ellipticity condition

$$\mu \geq \delta > 0, \quad \lambda + 2\mu \geq \delta, \quad \forall\, x \in \mathbb{R}^2. \tag{2.1}$$

The key to proving the global unique continuation property lies in arranging the coefficients nicely. We write (1.3) componentwisely

$$\partial_1(2\mu\partial_1 u_1) + \partial_2(\mu(\partial_1 u_2 + \partial_2 u_1)) + \partial_1(\lambda\nabla \cdot u) = 0 \tag{2.2}$$

and

$$\partial_1(\mu(\partial_1 u_2 + \partial_2 u_1)) + \partial_2(2\mu\partial_2 u_2) + \partial_2(\lambda\nabla \cdot u) = 0. \tag{2.3}$$

Let $v = \nabla \cdot u = \partial_1 u_1 + \partial_2 u_2$ and $w = \nabla \times u = \partial_1 u_2 - \partial_2 u_1$, then (2.2) is written as

$$\partial_1(2\mu\partial_1 u_1 + \lambda v) + \partial_2(2\mu\partial_2 u_1 + \mu w) = 0. \tag{2.4}$$

Similarly, (2.3) is equivalent to

$$\partial_1(2\mu\partial_1 u_2 - \mu w) + \partial_2(2\mu\partial_2 u_2 + \lambda v) = 0. \tag{2.5}$$

By taking advantage of the relation

$$\Delta u_1 = \partial_1 v - \partial_2 w \quad \text{and} \quad \Delta u_2 = \partial_2 v + \partial_1 w,$$

we obtain from (2.4) that

$$\partial_1((2\mu - 1)\partial_1 u_1 + (\lambda + 1)v) + \partial_2((2\mu - 1)\partial_2 u_1 + (\mu - 1)w) = 0 \tag{2.6}$$

and from (2.5) that

$$\partial_1((2\mu - 1)\partial_1 u_2 - (\mu - 1)w) + \partial_2((2\mu - 1)\partial_2 u_2 + (\lambda + 1)v) = 0. \tag{2.7}$$

Therefore, there exist u_1' and u_2' such that

$$\begin{cases} \partial_2 u_1' = (2\mu - 1)\partial_1 u_1 + (\lambda + 1)v, \\ -\partial_1 u_1' = (2\mu - 1)\partial_2 u_1 + (\mu - 1)w, \end{cases} \tag{2.8}$$

and

$$\begin{cases} \partial_2 u_2' = (2\mu - 1)\partial_1 u_2 - (\mu - 1)w, \\ -\partial_1 u_2' = (2\mu - 1)\partial_2 u_2 + (\lambda + 1)v. \end{cases} \tag{2.9}$$

Setting $f_1 = u_1 + iu_1'$ and $f_2 = u_2 + iu_2'$, (2.8) and (2.9) become

$$\begin{cases} \bar{\partial} f_1 = \sigma \overline{\partial f_1} + h, \\ \bar{\partial} f_2 = \sigma \overline{\partial f_2} + ih, \end{cases} \tag{2.10}$$

where

$$\sigma = \frac{1 - \mu}{\mu} \quad \text{and} \quad h = -\frac{\lambda + 1}{2\mu} v - i\frac{\mu - 1}{2\mu} w.$$

As usual, we define

$$\bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2), \quad \partial = \frac{1}{2}(\partial_1 - i\partial_2).$$

Using the obvious relations

$$\partial_1 u_1 = \frac{1}{2}(\bar{\partial} f_1 + \partial f_1 + \partial \bar{f}_1 + \overline{\partial f_1}), \quad \partial_1 u_2 = \frac{1}{2}(\bar{\partial} f_2 + \partial f_2 + \partial \bar{f}_2 + \overline{\partial f_2}),$$
$$\partial_2 u_1 = \frac{1}{2i}(\bar{\partial} f_1 - \partial f_1 - \partial \bar{f}_1 + \overline{\partial f_1}), \quad \partial_2 u_2 = \frac{1}{2i}(\bar{\partial} f_2 - \partial f_2 - \partial \bar{f}_2 + \overline{\partial f_2}),$$

we can compute

$$
\begin{aligned}
h &= -\frac{\lambda+1}{2\mu}v - i\frac{\mu-1}{2\mu}w \\
&= -\frac{\lambda+1}{4\mu}\left((\bar\partial f_1 + \partial f_1 + \partial\bar f_1 + \overline{\partial f_1}) - i(\bar\partial f_2 - \partial f_2 - \partial\bar f_2 + \overline{\partial f_2})\right) \\
&\quad - i\frac{\mu-1}{4\mu}\left((\bar\partial f_2 + \partial f_2 + \partial\bar f_2 + \overline{\partial f_2}) + i(\bar\partial f_1 - \partial f_1 - \partial\bar f_1 + \overline{\partial f_1})\right) \\
&= \left(\frac{\mu-\lambda-2}{4\mu}\right)\bar\partial f_1 + \left(\frac{-\lambda-\mu}{4\mu}\right)\partial f_1 + \left(\frac{-\lambda-\mu}{4\mu}\right)\partial\bar f_1 + \left(\frac{\mu-\lambda-2}{4\mu}\right)\overline{\partial f_1} \\
&\quad + i\left(\frac{\lambda-\mu+2}{4\mu}\right)\bar\partial f_2 + i\left(\frac{-\lambda-\mu}{4\mu}\right)\partial f_2 + i\left(\frac{-\lambda-\mu}{4\mu}\right)\partial\bar f_2 + i\left(\frac{\lambda-\mu+2}{4\mu}\right)\overline{\partial f_2}.
\end{aligned}
$$

For simplicity, let us denote

$$
\alpha = \frac{\mu-\lambda-2}{4\mu}, \quad \beta = \frac{-\lambda-\mu}{4\mu},
$$

then

$$
h = \alpha\bar\partial f_1 + \beta\partial f_1 + \beta\partial\bar f_1 + \alpha\overline{\partial f_1} - i\alpha\bar\partial f_2 + i\beta\partial f_2 + i\beta\partial\bar f_2 - i\alpha\overline{\partial f_2}
$$

and

$$
ih = i\alpha\bar\partial f_1 + i\beta\partial f_1 + i\beta\partial\bar f_1 + i\alpha\overline{\partial f_1} + \alpha\bar\partial f_2 - \beta\partial f_2 - \beta\partial\bar f_2 + \alpha\overline{\partial f_2}.
$$

Therefore, (2.10) is equivalent to

$$
\begin{aligned}
&\left(I_2 + \begin{pmatrix} -\alpha & i\alpha \\ -i\alpha & -\alpha \end{pmatrix}\right)\bar\partial\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} -\beta & -i\beta \\ -i\beta & \beta \end{pmatrix}\partial\begin{pmatrix} \bar f_1 \\ \bar f_2 \end{pmatrix} \\
&= \begin{pmatrix} \beta & i\beta \\ i\beta & -\beta \end{pmatrix}\partial\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} \sigma+\alpha & -i\alpha \\ i\alpha & \sigma+\alpha \end{pmatrix}\overline{\partial\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}},
\end{aligned} \tag{2.11}
$$

where I_n denotes the $n \times n$ unit matrix. Setting $\frac{\partial\bar f}{\bar\partial f} = 0$ if $\bar\partial f = 0$ and $\frac{\overline{\partial f}}{\partial f} = 0$ if $\partial f = 0$, (2.11) can be written as

$$
\begin{aligned}
&\left(I_2 + \begin{pmatrix} -\alpha & i\alpha \\ -i\alpha & -\alpha \end{pmatrix}\right)\bar\partial\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} -\beta & -i\beta \\ -i\beta & \beta \end{pmatrix}\begin{pmatrix} \frac{\partial\bar f_1}{\bar\partial f_1} & 0 \\ 0 & \frac{\partial\bar f_2}{\bar\partial f_2} \end{pmatrix}\bar\partial\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\
&= \begin{pmatrix} \beta & i\beta \\ i\beta & -\beta \end{pmatrix}\partial\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} \sigma+\alpha & -i\alpha \\ i\alpha & \sigma+\alpha \end{pmatrix}\begin{pmatrix} \frac{\overline{\partial f_1}}{\partial f_1} & 0 \\ 0 & \frac{\overline{\partial f_2}}{\partial f_2} \end{pmatrix}\partial\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.
\end{aligned} \tag{2.12}
$$

Finally, let U and V be two 2×2 matrices

$$
\begin{aligned}
U &= I_2 + \begin{pmatrix} -\alpha & i\alpha \\ -i\alpha & -\alpha \end{pmatrix} + \begin{pmatrix} -\beta & -i\beta \\ -i\beta & \beta \end{pmatrix}\begin{pmatrix} \frac{\partial\bar f_1}{\bar\partial f_1} & 0 \\ 0 & \frac{\partial\bar f_2}{\bar\partial f_2} \end{pmatrix}, \\
V &= \begin{pmatrix} \beta & i\beta \\ i\beta & -\beta \end{pmatrix} + \begin{pmatrix} \sigma+\alpha & -i\alpha \\ i\alpha & \sigma+\alpha \end{pmatrix}\begin{pmatrix} \frac{\overline{\partial f_1}}{\partial f_1} & 0 \\ 0 & \frac{\overline{\partial f_2}}{\partial f_2} \end{pmatrix},
\end{aligned}
$$

then (2.12) writes as

$$U\bar{\partial}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = V\,\partial\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{2.13}$$

It is clear that

$$\|U - I_2\|_{L^\infty} \le C(\|\alpha\|_{L^\infty} + \|\beta\|_{L^\infty}) \quad \text{and} \quad \|V\|_{L^\infty} \le C(\|\beta\|_{L^\infty} + \|\sigma\|_{L^\infty} + \|\alpha\|_{L^\infty}), \tag{2.14}$$

where C is an absolute constant. Hereafter, for any matrix-valued function $A(x) : \mathbb{C}^n \to \mathbb{C}^n$ with $x \in \mathbb{R}^2$, the norm $\|A\|_{L^\infty}$ is defined by

$$\|A\|_{L^\infty} = \sup_{x\in\mathbb{R}^2} \|A(x)\|,$$

where $\|\cdot\|$ is the usual matrix norm derived by treating $\mathbb{C}^n$ as an inner-product space.

Theorem 2.1 *There exists an* $\varepsilon > 0$ *such that if*

$$\|\mu - 1\|_{L^\infty} \le \varepsilon \quad \textit{and} \quad \|\lambda + 1\|_{L^\infty} \le \varepsilon, \tag{2.15}$$

then for any Lipschitz solution u *of* (1.3) *vanishing in the lower half plane, we must have* $u \equiv 0$.

Remark 2.2 *It is clear that if the Lamé coefficients* λ *and* μ *satisfy* (2.15)*, then the ellipticity condition* (2.1) *holds.*

Proof. Since u is Lipschitz and vanishes in the lower half plane, so does the vector-valued function

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

In view of the definitions of σ, α, β, if ε of (2.15) is sufficiently small, then all of them are sufficiently small as well. By (2.14), U is invertible and V is small, consequently, $\|U^{-1}V\|_{L^\infty} \le \varepsilon' \ll 1$. Therefore, from (2.13) we have

$$\bar{\partial}F = \Psi\partial F \quad \text{in} \quad \mathbb{C}, \tag{2.16}$$

where $\Psi = U^{-1}V$ and we have identified $\mathbb{R}^2$ as the complex plane $\mathbb{C}$. Note that here $F : \mathbb{C} \to \mathbb{C}^2$. (2.16) is a Beltrami system studied in [IVV02]. It was proved in [IVV02] that if $\|\Psi\|_{L^\infty}$ is sufficiently small and F vanishes in the lower half plane, then F is trivial, i.e., u is trivial. For the sake of completeness, we sketch the proof here. We refer to [IVV02, Section 7] for more details. Without loss of generality, we assume that F vanishes for $\Im z \le 1$. Define $G(z) = F(\sqrt{z})$. Then G satisfies

$$\bar{\partial}G = \frac{z}{|z|}\Psi(\sqrt{z})\partial G. \tag{2.17}$$

We can see that the differential of G, DG, lies in $L^p(\mathbb{C})$ for some $p > 4$ (see (7.13) in [IVV02]). In other words, we have

$$\|\bar{\partial}G\|_{L^p} \leq \varepsilon' \|\partial G\|_{L^p}. \tag{2.18}$$

Let S be the Beurling-Ahlfors transform, i.e., $S\bar{\partial} = \partial$. It is known from the Calderón-Zygmund theory that

$$\|S\|_{L^p \to L^p} \leq a_p \tag{2.19}$$

for some constant a_p, depending only on p (see [BJ08] for a more precise bound on a_p). Hence, (2.19) implies

$$\|\partial G\|_{L^p} \leq a_p \|\bar{\partial}G\|_{L^p}. \tag{2.20}$$

Combining (2.18) and (2.20), we conclude that $\|\bar{\partial}G\|_{L^p} = \|\partial G\|_{L^p} = 0$ provided $a_p\varepsilon' < 1$ for some $p > 4$. The proof of theorem then follows. $\square$

Now we turn to the general elasticity system (1.1). Assume that $a_{ijkl} \in L^\infty$ satisfies the ellipticity condition

$$a_{ijkl}(x)\xi_i\xi_l\rho_j\rho_k \geq \delta|\xi|^2|\rho|^2 \quad \forall\ \xi,\ \rho \in \mathbb{R}^2, \tag{2.21}$$

i.e., the Legendre-Hadamard condition. Due to the symmetry properties (1.2), for simplicity, we denote

$$\begin{cases} a_{1111} = a, a_{2222} = b, a_{1112} = a_{1211} = a_{2111} = a_{1121} = c, \\ a_{1122} = a_{2211} = d, a_{1212} = a_{2112} = a_{1221} = a_{2121} = e, \\ a_{1222} = a_{2122} = a_{2212} = a_{2221} = g. \end{cases}$$

Componentwise, (1.1) is written as

$$\begin{cases} \partial_1(a\partial_1 u_1 + c\partial_1 u_2 + c\partial_2 u_1 + d\partial_2 u_2) + \partial_2(c\partial_1 u_1 + e\partial_1 u_2 + e\partial_2 u_1 + g\partial_2 u_2) = 0, \\ \partial_1(c\partial_1 u_1 + e\partial_1 u_2 + e\partial_2 u_1 + g\partial_2 u_2) + \partial_2(d\partial_1 u_1 + g\partial_1 u_2 + g\partial_2 u_1 + b\partial_2 u_2) = 0. \end{cases} \tag{2.22}$$

For our purpose, we will express (2.22) as

$$\begin{cases} \partial_1((a-d-1)\partial_1 u_1 + (d+1)v + c\partial_1 u_2 + c\partial_2 u_1) \\ \quad + \partial_2((2e-1)\partial_2 u_1 + (e-1)w + c\partial_1 u_1 + g\partial_2 u_2) = 0, \\ \partial_1((2e-1)\partial_1 u_2 - (e-1)w + c\partial_1 u_1 + g\partial_2 u_2) \\ \quad + \partial_2((b-d-1)\partial_2 u_2 + (d+1)v + g\partial_1 u_2 + g\partial_2 u_1) = 0. \end{cases} \tag{2.23}$$

Comparing (2.23) with (2.6), (2.7), it is not difficult to see that (2.23) can be transformed to (2.16) with similar smallness condition on Ψ if $|a-d-2| \ll 1, |b-d-2| \ll 1, |d+1| \ll 1, |e-1| \ll 1, |c| \ll 1, |g| \ll 1$. Therefore, we can prove that

Theorem 2.3 *There exists $\varepsilon > 0$ such that if*

$$\begin{cases} \|a-d-2\|_{L^\infty} \leq \varepsilon, \|b-d-2\|_{L^\infty} \leq \varepsilon, \|d+1\|_{L^\infty} \leq \varepsilon, \\ \|e-1\|_{L^\infty} \leq \varepsilon, \|c\| \leq \varepsilon, \|g\|_{L^\infty} \leq \varepsilon, \end{cases} \tag{2.24}$$

then if u is a Lipschitz function solving (1.1) *and vanishes in the lower half plane, then u vanishes identically.*

Remark 2.4 *Under the assumptions* (2.24), (1.1) *is a slightly perturbed system of the Lamé system with λ, μ satisfying* (2.15)*. Therefore, the ellipticity condition* (2.21) *holds.*

3 Anisotropic system with regular coefficients

One may wonder if $|d+1| \ll 1$ and $|e-1| \ll 1$ in Theorem 2.3 can be replaced by $|d+k_0| \ll 1$ and $|e-k_0| \ll 1$ for $k_0 \neq 1$. For measurable coefficients, it is not possible since the requirement of $|e-k_0| \ll 1$ and $2e-k_0 \sim 1$ will force $k_0 = 1$. However, if a, b, d, e are Lipschitz, we can extend Theorem 2.3 to a larger class of system. Let k_0 be any fixed constant. Similarly to (2.23), we obtain that

$$\begin{cases} \partial_1((a-d-k_0)\partial_1 u_1 + (d+k_0)v + c\partial_1 u_2 + c\partial_2 u_1) \\ \quad + \partial_2((2e-k_0)\partial_2 u_1 + (e-k_0)w + c\partial_1 u_1 + g\partial_2 u_2) = 0, \\ \partial_1((2e-k_0)\partial_1 u_2 - (e-k_0)w + c\partial_1 u_1 + g\partial_2 u_2) \\ \quad + \partial_2((b-d-k_0)\partial_2 u_2 + (d+k_0)v + g\partial_1 u_2 + g\partial_2 u_1) = 0. \end{cases} \tag{3.1}$$

Denote $\tilde{a} = a-d-k_0, \tilde{e} = 2e-k_0, \tilde{b} = b-d-k_0$. Suppose that $\tilde{a} \neq 0$ and $\tilde{b} \neq 0$. Then (3.1) is equivalent to

$$\begin{cases} \partial_1(\partial_1(\tilde{a}u_1) - \partial_1\tilde{a}u_1 + (d+k_0)v + c\partial_1 u_2 + c\partial_2 u_1) \\ \quad + \partial_2(\tilde{e}\tilde{a}^{-1}(\partial_2(\tilde{a}u_1) - \partial_2\tilde{a}u_1) + (e-k_0)w + c\partial_1 u_1 + g\partial_2 u_2) = 0, \\ \partial_1(\tilde{e}\tilde{b}^{-1}(\partial_1(\tilde{b}u_2) - \partial_1\tilde{b}u_2) - (e-k_0)w + c\partial_1 u_1 + g\partial_2 u_2) \\ \quad + \partial_2(\partial_2(\tilde{b}u_2) - \partial_2\tilde{b}u_2 + (d+k_0)v + g\partial_1 u_2 + g\partial_2 u_1) = 0. \end{cases} \tag{3.2}$$

We set $\tilde{u}_1 = \tilde{a}u_1, \tilde{u}_2 = \tilde{b}u_2$, then (3.2) becomes

$$\begin{cases} \partial_1(\partial_1(\tilde{u}_1) - \partial_1\tilde{a}\tilde{a}^{-1}\tilde{u}_1 + (d+k_0)v + c\partial_1(\tilde{b}^{-1}\tilde{u}_2) + c\partial_2(\tilde{a}^{-1}\tilde{u}_1)) \\ \quad + \partial_2(\tilde{e}\tilde{a}^{-1}(\partial_2(\tilde{u}_1) - \partial_2\tilde{a}\tilde{a}^{-1}\tilde{u}_1) + (e-k_0)w + c\partial_1(\tilde{a}^{-1}\tilde{u}_1) + g\partial_2(\tilde{b}^{-1}\tilde{u}_2)) = 0, \\ \partial_1(\tilde{e}\tilde{b}^{-1}(\partial_1(\tilde{u}_2) - \partial_1\tilde{b}\tilde{b}^{-1}\tilde{u}_2) - (e-k_0)w + c\partial_1(\tilde{a}^{-1}\tilde{u}_1) + g\partial_2(\tilde{b}^{-1}\tilde{u}_2)) \\ \quad + \partial_2(\partial_2(\tilde{u}_2) - \partial_2\tilde{b}\tilde{b}^{-1}\tilde{u}_2 + (d+k_0)v + g\partial_1(\tilde{b}^{-1}\tilde{u}_2) + g\partial_2(\tilde{a}^{-1}\tilde{u}_1)) = 0. \end{cases} \tag{3.3}$$

We then express

$$\begin{cases} v = \partial_1 u_1 + \partial_2 u_2 = \partial_1\tilde{a}^{-1}\tilde{u}_1 + \tilde{a}^{-1}\partial_1\tilde{u}_1 + \partial_2\tilde{b}^{-1}\tilde{u}_2 + \tilde{b}^{-1}\partial_2\tilde{u}_2 \\ w = \partial_1 u_2 - \partial_2 u_1 = \partial_1\tilde{b}^{-1}\tilde{u}_2 + \tilde{b}^{-1}\partial_1\tilde{u}_2 - \partial_2\tilde{a}^{-1}\tilde{u}_1 - \tilde{a}^{-1}\partial_2\tilde{u}_1. \end{cases}$$

Theorem 3.1 *Let $k_0 > 0$. Assume that a, b, d, e are Lipschitz and c, g are measurable. Moreover, suppose that a, b, d, e are constants in $\mathbb{R}^2 \setminus K$, where K is compact set. Then there exists an $\varepsilon = \varepsilon(k_0, K) > 0$ such that if*

$$\begin{cases} \|\nabla(a-d)\|_{L^\infty} \le \varepsilon, \ \|\nabla(b-d)\|_{L^\infty} \le \varepsilon, \ \|a-d-2e\|_{L^\infty} \le \varepsilon, \ \|b-d-2e\|_{L^\infty} \le \varepsilon, \\ \|d+k_0\|_{L^\infty} \le \varepsilon, \ \|e-k_0\|_{L^\infty} \le \varepsilon, \ \|c\|_{L^\infty} \le \varepsilon, \ \|g\|_{L^\infty} \le \varepsilon, \end{cases} \tag{3.4}$$

then if u vanishes in the lower half plane, then u is identically zero.

Proof. We first note that when ε, depending on k_0, is sufficiently small, $\tilde{a}$ and $\tilde{b}$ are strictly positive. Let $f_1 = \tilde{u}_1 + \tilde{u}_1'$ and $f_2 = \tilde{u}_2 + i\tilde{u}_2'$, where $\tilde{u}_1'$ and $\tilde{u}_2'$ are conjugate functions of $\tilde{u}_1$ and $\tilde{u}_2$ defined as above. In view of (3.4), (3.3) is reduced to

$$\bar{\partial} F = \tilde{\Psi} \partial F + HF + \tilde{H} \bar{F}, \tag{3.5}$$

where $\|\tilde{\Psi}\|_{L^\infty} \le \varepsilon'$, $\|H\|_{L^\infty} \le \varepsilon'$, $\|\tilde{H}\|_{L^\infty} \le \varepsilon'$, and H, $\tilde{H}$ are supported in K. Note that $\varepsilon' \to 0$ as $\varepsilon \to 0$. As before, let $G(z) = F(\sqrt{z})$, then G satisfies

$$\bar{\partial} G = \frac{z}{|z|} \tilde{\Psi}(\sqrt{z}) \partial G + H(\sqrt{z}) G + \tilde{H}(\sqrt{z}) \bar{G}.$$

By the Poincaré inequality, we have that

$$\|HG\|_{L^p} + \|\tilde{H}\bar{G}\|_{L^p} \le \varepsilon' C (\|\bar{\partial} G\|_{L^p} + \|\partial G\|_{L^p}) \tag{3.6}$$

for $p \ge 2$, where C depends on K (and p). Using (3.6), we have from (3.5) that

$$\|\bar{\partial} G\|_{L^p} \le \varepsilon'' \|\partial G\|_{L^p}$$

with $\varepsilon'' \to 0$ as $\varepsilon \to 0$. Next, using the same arguments as in the proof of Theorem 2.1, the result follows. $\square$

Remark 3.2 *From the ellipticity condition* (2.1) *for isotropic media, it is readily seen that if $k_0 > 0$ and ε is sufficiently small, then the ellipticity condition* (2.21) *is satisfied.*

4 Counterexample to unique continuation

In this section we will construct a counterexample to the unique continuation property, which vanishes in the lower half plane, for second order elliptic systems with measurable coefficients. Precisely, we consider

$$\partial_j (a_{ijkl}(x) \partial_k u_l) = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{4.1}$$

where the coefficients a_{ijkl} do not necessarily satisfy the symmetry conditions (1.2). For simplicity, we use the following short-hand notations:

$$11 \to 1,\ 12 \to 2,\ 21 \to 3,\ 22 \to 4,$$

i.e.,

$$a_{1111} = a_{11}, a_{1112} = a_{12}, a_{1121} = a_{13}, a_{1122} = a_{14}, \cdots \text{ etc.}$$

So, the system (4.1) is written as

$$\begin{cases} \partial_1(a_{11}\partial_1 u_1 + a_{12}\partial_1 u_2 + a_{13}\partial_2 u_1 + a_{14}\partial_2 u_2) \\ \qquad + \partial_2(a_{21}\partial_1 u_1 + a_{22}\partial_1 u_2 + a_{23}\partial_2 u_1 + a_{24}\partial_2 u_2) = 0, \\ \partial_1(a_{31}\partial_1 u_1 + a_{32}\partial_1 u_2 + a_{33}\partial_2 u_1 + a_{34}\partial_2 u_2) \\ \qquad + \partial_2(a_{41}\partial_1 u_1 + a_{42}\partial_1 u_2 + a_{43}\partial_2 u_1 + a_{44}\partial_2 u_2) = 0. \end{cases} \tag{4.2}$$

As before, we can find v_1 and v_2 such that

$$\begin{cases} \partial_2 v_1 = a_{11}\partial_1 u_1 + a_{13}\partial_2 u_1 + a_{12}\partial_1 u_2 + a_{14}\partial_2 u_2, \\ -\partial_1 v_1 = a_{21}\partial_1 u_1 + a_{23}\partial_2 u_1 + a_{22}\partial_1 u_2 + a_{24}\partial_2 u_2, \end{cases} \tag{4.3}$$

and

$$\begin{cases} \partial_2 v_2 = a_{32}\partial_1 u_2 + a_{34}\partial_2 u_2 + a_{31}\partial_1 u_1 + a_{33}\partial_2 u_1, \\ -\partial_1 v_2 = a_{42}\partial_1 u_2 + a_{44}\partial_2 u_2 + a_{41}\partial_1 u_1 + a_{43}\partial_2 u_1. \end{cases} \tag{4.4}$$

Here we will use a different reduction from the one used in Section 2. The method is inspired by Bojarski's work [Bo57]. Denote

$$\alpha_1 = \frac{(a_{11} + a_{23}) + i(a_{21} - a_{13})}{2}, \quad \beta_1 = \frac{(a_{11} - a_{23}) + i(a_{21} + a_{13})}{2},$$
$$\zeta_1 = a_{12} + i a_{22}, \quad \eta_1 = a_{14} + i a_{24}.$$

Let $f_1 = u_1 + iv_1$ and $f_2 = u_2 + iv_2$, then we can compute that

$$\begin{aligned} 0 &= (1+\alpha_1)\bar\partial f_1 + \beta_1 \partial \overline{f_1} + \beta_1 \partial f_1 - (1-\alpha_1)\overline{\partial f_1} + \zeta_1 \partial_1 u_2 + \eta_1 \partial_2 u_2 \\ &= (1+\alpha_1)\bar\partial f_1 + \beta_1 \partial \overline{f_1} + \beta_1 \partial f_1 - (1-\alpha_1)\overline{\partial f_1} + \frac{\zeta_1}{2}(\bar\partial f_2 + \partial f_2 + \partial\overline{f_2} + \overline{\partial f_2}) \\ &\quad + \frac{\eta_1}{2i}(\bar\partial f_2 - \partial f_2 - \partial \overline{f_2} + \overline{\partial f_2}) \\ &= (1+\alpha_1)\bar\partial f_1 + \beta_1 \partial \overline{f_1} + \beta_1 \partial f_1 - (1-\alpha_1)\overline{\partial f_1} + (\frac{\zeta_1}{2} + \frac{\eta_1}{2i})\bar\partial f_2 + (\frac{\zeta_1}{2} - \frac{\eta_1}{2i})\partial f_2 \\ &\quad + (\frac{\zeta_1}{2} - \frac{\eta_1}{2i})\partial\overline{f_2} + (\frac{\zeta_1}{2} + \frac{\eta_1}{2i})\overline{\partial f_2}. \end{aligned} \tag{4.5}$$

Likewise, we denote

$$\alpha_2 = \frac{(a_{32}+a_{44})+i(a_{42}-a_{34})}{2}, \quad \beta_2 = \frac{(a_{32}-a_{44})+i(a_{42}+a_{34})}{2},$$
$$\zeta_2 = a_{31}+ia_{41}, \quad \eta_2 = a_{33}+ia_{43},$$

then we obtain

$$\begin{aligned} 0 = (1+\alpha_2)\bar\partial f_2 + \beta_2\partial\overline{f_2} + \beta_2\partial f_2 - (1-\alpha_2)\overline{\partial f_2} + (\frac{\zeta_2}{2}+\frac{\eta_2}{2i})\bar\partial f_1 + (\frac{\zeta_2}{2}-\frac{\eta_2}{2i})\partial f_1 \\ + (\frac{\zeta_2}{2}-\frac{\eta_2}{2i})\partial\overline{f_1} + (\frac{\zeta_2}{2}+\frac{\eta_2}{2i})\overline{\partial f_1}. \end{aligned} \tag{4.6}$$

Putting (4.5) and (4.6) in matrix form gives

$$A\bar\partial F + B\partial\overline{F} + C\partial F + D\overline{\partial F} = 0 \quad \text{in} \quad \mathbb{C}, \tag{4.7}$$

where $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and

$$\begin{aligned} A &= \begin{pmatrix} 1+\alpha_1 & \frac{\zeta_1}{2}-i\frac{\eta_1}{2} \\ \frac{\zeta_2}{2}-i\frac{\eta_2}{2} & 1+\alpha_2 \end{pmatrix}, \quad B = \begin{pmatrix} \beta_1 & \frac{\zeta_1}{2}+i\frac{\eta_1}{2} \\ \frac{\zeta_2}{2}+i\frac{\eta_2}{2} & \beta_2 \end{pmatrix}, \\ C &= \begin{pmatrix} \beta_1 & \frac{\zeta_1}{2}+i\frac{\eta_1}{2} \\ \frac{\zeta_2}{2}+i\frac{\eta_2}{2} & \beta_2 \end{pmatrix} (=B), \quad D = \begin{pmatrix} -1+\alpha_1 & \frac{\zeta_1}{2}-i\frac{\eta_1}{2} \\ \frac{\zeta_2}{2}-i\frac{\eta_2}{2} & -1+\alpha_2 \end{pmatrix}. \end{aligned} \tag{4.8}$$

Note that $D = A - 2I_2$. Conversely, it is easy to see that, given any 2×2 complex-valued matrices A, B, C, D satisfying $B = C$ and $D = A - 2I_2$ and (4.7) with $F = \begin{pmatrix} u_1+iv_1 \\ u_2+iv_2 \end{pmatrix}$, then, writing A, B, C, D as in (4.8), we can find real numbers $a_{11}, a_{12}, \cdots, a_{44}$ such that (4.3), (4.4) hold, and hence (4.2) is satisfied.

It was proved in [IVV02] that there exists a 2×2 complex-valued matrix $Q \in L^\infty(\mathbb{C})$ with

$$\|Q\|_{L^\infty(\mathbb{C})} \le \kappa < 1 \tag{4.9}$$

and a nontrivial Lipschitz function $\tilde F : \mathbb{C} \to \mathbb{C}^2$ vanishing in the lower half plane of $\mathbb{C}$ such that

$$\bar\partial\tilde F + Q\partial\tilde F = 0 \quad \text{in} \quad \mathbb{C}. \tag{4.10}$$

Adding $A \times$ (4.10) and $B \times \overline{(4.10)}$, for any A, B, gives

$$A\bar\partial\tilde F + B\partial\overline{\tilde F} + AQ\partial\tilde F + B\overline{Q}\,\overline{\partial\tilde F} = 0 \quad \text{in} \quad \mathbb{C}. \tag{4.11}$$

Comparing (4.7) with (4.11), we hope to find A, B, C, D satisfying

$$B = C = AQ, \quad D = B\overline{Q}, \quad D = A - 2I_2. \tag{4.12}$$

To fulfill (4.12), we begin with

$$A - 2I_2 = D = AQ\overline{Q},$$

which implies

$$A(I_2 - Q\overline{Q}) = 2I_2. \tag{4.13}$$

In view of (4.9), we have that $\|Q\overline{Q}\|_{L^\infty(\mathbb{C})} \leq \kappa^2 < 1$ and hence $I_2 - Q\overline{Q}$ is invertible. In other words, (4.13) gives

$$A = 2(I_2 - Q\overline{Q})^{-1}.$$

Once A is determined, we can find C and, of course, B. Hence the relations in (4.12) hold. Finally, in view of the definitions of $\alpha_j, \beta_j, \zeta_j, \eta_j, j = 1, 2$, there exists a unique fourth rank-tensor $(a_{ijkl}(x))$ producing A, B, C, D which were determined above.

With such A, B, C, D obtained above, there exists a nontrivial solution $F : \mathbb{C} \to \mathbb{C}^2$, i.e., $F = \tilde{F}$, vanishing in the lower half plane of $\mathbb{C}$ and satisfying (4.7) (and hence, (4.11)), i.e.,

$$A\bar{\partial}F + AQ\partial\overline{F} + AQ\partial F + AQ\overline{Q}\,\overline{\partial F} = 0 \quad \text{in} \quad \mathbb{C}. \tag{4.14}$$

As mentioned above, (4.14) is equivalent to the second order system (4.2) with corresponding coefficients $(a_{ijkl}(x))$. Now we would like to verify that this second order system is elliptic. The meaning of ellipticity will be specified later. We first show that $L_0F := \bar{\partial}F + Q\partial F$ is equivalent to a first order uniformly elliptic system. Let us denote

$$F = \begin{pmatrix} u_1 + iv_1 \\ u_2 + iv_2 \end{pmatrix},$$

then

$$2\bar{\partial}F = \begin{pmatrix} (\partial_1 u_1 - \partial_2 v_1) + i(\partial_1 v_1 + \partial_2 u_1) \\ (\partial_1 u_2 - \partial_2 v_2) + i(\partial_1 v_2 + \partial_2 u_2) \end{pmatrix}$$

and

$$2\partial F = \begin{pmatrix} (\partial_1 u_1 + \partial_2 v_1) + i(\partial_1 v_1 - \partial_2 u_1) \\ (\partial_1 u_2 + \partial_2 v_2) + i(\partial_1 v_2 - \partial_2 u_2) \end{pmatrix}.$$

Let $Q = Q_r + iQ_i$, then $2L_0F$ can be put into the following equivalent system

$$L_1 \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} := \begin{pmatrix} I_2 + Q_r & -Q_i \\ Q_i & I_2 + Q_r \end{pmatrix} \partial_1 \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} Q_i & -I_2 + Q_r \\ I_2 - Q_r & Q_i \end{pmatrix} \partial_2 \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix}, \tag{4.15}$$

i.e., $2L_0F = G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} w_1 + iz_1 \\ w_2 + iz_2 \end{pmatrix}$ is equivalent to

$$L_1 \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ z_1 \\ z_2 \end{pmatrix}.$$

For simplicity, we denote

$$R = \begin{pmatrix} I_2 + Q_r & -Q_i \\ Q_i & I_2 + Q_r \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} Q_i & -I_2 + Q_r \\ I_2 - Q_r & Q_i \end{pmatrix}.$$

Now we want to show that (4.15) is uniformly elliptic, i.e.,

$$\det(\alpha R + \beta S) \geq c(\alpha^2 + \beta^2)^2, \quad \forall \ z \in \mathbb{C}, \ (\alpha, \beta) \in \mathbb{R}^2 \neq 0, \tag{4.16}$$

where $c = c(\kappa) > 0$. To prove (4.16), we first observe that

$$S = \begin{pmatrix} Q_i & -I_2 + Q_r \\ I_2 - Q_r & Q_i \end{pmatrix} = \begin{pmatrix} I_2 - Q_r & Q_i \\ -Q_i & I_2 - Q_r \end{pmatrix} J$$

where

$$J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}.$$

Therefore, we obtain that

$$\alpha R + \beta S = \alpha(I_4 + E) + \beta(I_4 - E)J = (\alpha I_4 + \beta J) + E(\alpha I_4 - \beta J), \tag{4.17}$$

where

$$E = \begin{pmatrix} Q_r & -Q_i \\ Q_i & Q_r \end{pmatrix}.$$

From (4.9), we have that for $E : \mathbb{R}^4 \to \mathbb{R}^4$

$$\|E\|_{L^\infty(\mathbb{R}^4)} \leq \kappa. \tag{4.18}$$

It is easy to see that

$$\|(\alpha I_4 + \beta J)Z\| = \|(\alpha I_4 - \beta J)Z\| = \sqrt{\alpha^2 + \beta^2} \ \|Z\|, \quad \forall \ Z \in \mathbb{R}^4. \tag{4.19}$$

Combining (4.18) and (4.19) gives

$$\det(\alpha R + \beta S) = \det(\alpha I_4 + \beta J)\det(I_4 + (\alpha I_4 + \beta J)^{-1} E(\alpha I_4 - \beta J)) \geq c(\alpha^2 + \beta^2)^2$$

with $c = c(\kappa)$ and (4.16) is proved.

Now we want to consider

$$2(\bar{\partial} F + Q \partial F + Q(\partial \overline{F} + \overline{Q} \ \overline{\partial F})) = 2L_0 F + 2Q\overline{L_0 F}. \tag{4.20}$$

It is easy to see that

$$2\overline{L_0 F} = \begin{pmatrix} w_1 - iz_1 \\ w_2 - iz_2 \end{pmatrix},$$

which is equivalent to

$$\hat{I}L_1 \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = \hat{I} \begin{pmatrix} w_1 \\ w_2 \\ z_1 \\ z_2 \end{pmatrix},$$

where

$$\hat{I} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$

Consequently, (4.20) can be written as

$$(R + E\hat{I}R)\partial_1 \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} + (S + E\hat{I}S)\partial_2 \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix}.$$

It is clear that

$$\begin{aligned} \det(\alpha(R + E\hat{I}R) + \beta(S + E\hat{I}S)) &= \det([\alpha R + \beta S] + E\hat{I}[\alpha R + \beta S]) \\ &= \det(\alpha R + \beta S) \cdot \det(I_4 + E\hat{I}) \\ &\geq c(\alpha^2 + \beta^2)^2. \end{aligned}$$

Finally, let us denote $A = A_r + iA_i$ and

$$\hat{A} = \begin{pmatrix} A_r & -A_i \\ A_i & A_r \end{pmatrix},$$

then

$$A\bar{\partial}F + AQ\partial\overline{F} + AQ\partial F + AQ\overline{Q}\,\overline{\partial F} = 0 \tag{4.21}$$

is equivalent to the first order system

$$\hat{A}(R + E\hat{I}R)\partial_1 \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} + \hat{A}(S + E\hat{I}S)\partial_2 \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = 0. \tag{4.22}$$

Since $\det\hat{A} \geq c > 0$, we immediately obtain that

$$\det(\alpha\hat{A}(R + E\hat{I}R) + \beta\hat{A}(S + E\hat{I}S)) \geq c(\alpha^2 + \beta^2)^2. \tag{4.23}$$

We would like to remind the reader that (4.21) is equivalent to (4.3), (4.4) (and (4.2)).

Now we return to the system (4.3), (4.4), i.e.,

$$\begin{cases} \partial_2 v_1 = a_{11}\partial_1 u_1 + a_{13}\partial_2 u_1 + a_{12}\partial_1 u_2 + a_{14}\partial_2 u_2, \\ -\partial_1 v_1 = a_{21}\partial_1 u_1 + a_{23}\partial_2 u_1 + a_{22}\partial_1 u_2 + a_{24}\partial_2 u_2, \end{cases}$$

and

$$\begin{cases} \partial_2 v_2 = a_{32}\partial_1 u_2 + a_{34}\partial_2 u_2 + a_{31}\partial_1 u_1 + a_{33}\partial_2 u_1, \\ -\partial_1 v_2 = a_{42}\partial_1 u_2 + a_{44}\partial_2 u_2 + a_{41}\partial_1 u_1 + a_{43}\partial_2 u_1. \end{cases}$$

We put this system as

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 1 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 1 \end{pmatrix} \partial_1 \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} a_{13} & a_{14} & -1 & 0 \\ a_{23} & a_{24} & 0 & 0 \\ a_{33} & a_{34} & 0 & -1 \\ a_{43} & a_{44} & 0 & 0 \end{pmatrix} \partial_2 \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = 0, \tag{4.24}$$

which is equivalent to (4.22). From (4.23), we have that

$$\begin{aligned} & c(\alpha^2+\beta^2)^2 \\ & \le \det(\alpha \hat{A}(R+E\hat{I}R) + \beta\hat{A}(S+E\hat{I}S)) \\ & = \det \left(\alpha \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 1 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} a_{13} & a_{14} & -1 & 0 \\ a_{23} & a_{24} & 0 & 0 \\ a_{33} & a_{34} & 0 & -1 \\ a_{43} & a_{44} & 0 & 0 \end{pmatrix} \right) \\ & = -\det \left(\alpha \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{21} & a_{22} & 1 & 0 \\ a_{41} & a_{42} & 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} a_{13} & a_{14} & -1 & 0 \\ a_{33} & a_{34} & 0 & -1 \\ a_{23} & a_{24} & 0 & 0 \\ a_{43} & a_{44} & 0 & 0 \end{pmatrix} \right) \\ & = \det \begin{pmatrix} a_{12}\alpha + a_{14}\beta & a_{11}\alpha + a_{13}\beta & -\beta & 0 \\ a_{32}\alpha + a_{34}\beta & a_{31}\alpha + a_{33}\beta & 0 & -\beta \\ a_{22}\alpha + a_{24}\beta & a_{21}\alpha + a_{23}\beta & \alpha & 0 \\ a_{42}\alpha + a_{44}\beta & a_{41}\alpha + a_{43}\beta & 0 & \alpha \end{pmatrix} \\ & = \det \begin{pmatrix} a_{12}\alpha^2 + a_{14}\alpha\beta + a_{22}\alpha\beta + a_{24}\beta^2 & a_{11}\alpha^2 + a_{13}\alpha\beta + a_{21}\alpha\beta + a_{23}\beta^2 \\ a_{32}\alpha^2 + a_{34}\alpha\beta + a_{42}\alpha\beta + a_{44}\beta^2 & a_{31}\alpha^2 + a_{33}\alpha\beta + a_{41}\alpha\beta + a_{43}\beta^2 \end{pmatrix}. \end{aligned} \tag{4.25}$$

It follows from (4.25) that for any $\xi = (\xi_1, \xi_2) \neq 0$, the 2×2 matrix $(\sum_{j,k} a_{ijkl}(z)\xi_j\xi_k)$ satisfies

$$|\det(\sum_{j,k} a_{ijkl}(z)\xi_j\xi_k)| \ge c|\xi|^4, \ \ \forall \ \ z \in \mathbb{C}. \tag{4.26}$$

In summary, we have shown that

Theorem 4.1 *There exists a nontrivial vector-valued function* $u = (u_1, u_2)^T : \mathbb{R}^2 \to \mathbb{R}^2$ *vanishing in the lower half plane solving a second order uniformly elliptic system* (4.1)*, in the sense of* (4.26)*, with essentially bounded coefficients.*

To prove Theorem 4.1, we used a reduction different from the one given in Section 2. It is natural to investigate whether the reduction used here can be applied to prove positive results stated in Section 2. We only discuss the Lamé system. Comparing (2.2), (2.3) and (4.2) implies

$$\begin{cases} a_{11} = \lambda + 2\mu,\ a_{12} = a_{13} = 0,\ a_{14} = \lambda, \\ a_{21} = a_{24} = 0,\ a_{22} = a_{23} = \mu, \\ a_{31} = a_{34} = 0,\ a_{32} = a_{33} = \mu, \\ a_{41} = \lambda,\ a_{42} = a_{43} = 0,\ a_{44} = \lambda + 2\mu. \end{cases}$$

By the definitions, we see that

$$\begin{cases} \alpha_1 = \dfrac{\lambda + 3\mu}{2},\ \beta_1 = \dfrac{\lambda + \mu}{2},\ \zeta_1 = i\mu,\ \eta_1 = \lambda, \\ \alpha_2 = \dfrac{\lambda + 3\mu}{2},\ \beta_2 = -\dfrac{\lambda + \mu}{2},\ \zeta_2 = i\lambda,\ \eta_2 = \mu, \end{cases}$$

and thus,

$$A = \begin{pmatrix} 1 + \frac{\lambda+3\mu}{2} & \frac{i}{2}(\mu - \lambda) \\ \frac{i}{2}(\lambda - \mu) & 1 + \frac{\lambda+3\mu}{2} \end{pmatrix}, \quad B = C = \begin{pmatrix} \frac{\lambda+\mu}{2} & \frac{i}{2}(\mu + \lambda) \\ \frac{i}{2}(\lambda + \mu) & -\frac{\lambda+\mu}{2} \end{pmatrix},$$

$$D = \begin{pmatrix} -1 + \frac{\lambda+3\mu}{2} & \frac{i}{2}(\mu - \lambda) \\ \frac{i}{2}(\lambda - \mu) & -1 + \frac{\lambda+3\mu}{2} \end{pmatrix}.$$

Now if $\mu \approx 1$ and $\lambda \approx -1$ as in Theorem 2.1, then $B \approx 0$, $C \approx 0$, but

$$A \approx \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix},\ D = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

In other words, (4.7) corresponding to the Lamé system can not be put into the form

$$\bar{\partial} F = \Psi \partial F$$

with $\|\Psi\|_{L^\infty} \ll 1$.

On the other hand, if the coefficients (a_{pq}) of (4.2) satisfy

$$\begin{cases} \|a_{pq} - 1\|_{L^\infty} \le \varepsilon \ \text{ for } \ pq = 11, 23, 32, 44, \\ \|a_{pq}\|_{L^\infty} \le \varepsilon \ \text{ for all other } pq's \end{cases} \tag{4.27}$$

with a sufficiently small ε, then

$$\|A - 2I_2\|_{L^\infty} \le c\varepsilon,\ \|B\|_{L^\infty} = \|C\|_{L^\infty} \le c\varepsilon,\ \|D\|_{L^\infty} \le c\varepsilon$$

for some constant c. For this case, we can prove that the global unique continuation property holds as in the proof of Theorem 2.3. It is not hard to see that the second order system (4.2) with coefficients satisfying (4.27) is elliptic in the sense of (4.26). In fact, we can even show that $(a_{pq}) = (a_{ijkl})$ satisfies the strong convexity condition (1.6) (and, of course, the Legendre-Hadamard condition (2.21)) provided ε is small. To see this, it suffices to consider $a_{11} = a_{1111} = 1$, $a_{23} = a_{1221} = 1$, $a_{32} = a_{2112} = 1$, $a_{44} = a_{2222} = 1$, and all other a_{pq}'s are zero. Then we have that for any 2×2 matrix $\xi = (\xi_k^l)$

$$a_{ijkl}\xi_k^l\xi_j^i = a_{1111}\xi_1^1\xi_1^1 + a_{1221}\xi_2^1\xi_2^1 + a_{2112}\xi_1^2\xi_1^2 + a_{2222}\xi_2^2\xi_2^2 = |\xi|^2,$$

which implies (1.6) for small ε. The class of second order elliptic systems (4.2) satisfying (4.27) contains a special class of hyperelastic materials, where only the major symmetry property $a_{ijkl} = a_{klij}$ holds.

A counterexample to the strong unique continuation for (2.16) was constructed in [CP11] (see also related article [Ro09]). The counterexample given in [CP11] shows that there exists a nontrivial function F vanishing at 0 to infinite order satisfying

$$\bar{\partial}F + Q\partial F = 0,$$

where $Q(x) \in \mathbb{C}^{2\times 2}$ is continuous and vanishes at 0 to infinite order as well. Based on this example, using the same framework as above, we can construct a counterexample to strong unique continuation for second order elliptic systems with continuous coefficients in the plane. Observe that for the extreme case $Q = 0$, we have $A = 2I$ and $B = C = D = 0$. Consequently, we see that

$$a_{11} = a_{23} = a_{32} = a_{44} = 1$$

and all other a_{pq}'s are zero. Therefore, when x is near 0, Q is sufficiently small, which is exactly the case we discussed in (4.27). In other words, for the second order elliptic system with coefficients satisfying (4.27) and the strong convexity condition (1.6), the global unique continuation property holds, in spite of the fact that there are examples showing that the strong unique continuation property fails. Furthermore, we want to point out that the counterexample to the strong unique continuation for (4.1) we constructed is a small perturbation of the Laplacian Δ near the origin. In Section 2 we have shown that the Lamé system with $\lambda \approx -1$ and $\mu \approx 1$ can be written as a small perturbation of the Laplacian. Therefore, this counterexample strongly suggests that the Lamé system with measurable coefficients, even when $\lambda \approx -1$ and $\mu \approx 1$, does not possess the strong unique continuation property. Moreover, this example or an earlier example constructed in [Ali80] also suggest that the strong unique continuation property for the anisotropic elasticity system even with continuous coefficients is most likely not true.

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