VARIETIES WITH $P_3(X) = 4$ AND $q(X) = \dim(X)$

ALFRED JUNGKAI CHEN AND CHRISTOPHER D. HACON

Abstract. We classify varieties with $P_3(X) = 4$ and $q(X) = \dim(X)$.

1. Introduction

Let $X$ be a smooth complex projective variety. When $\dim(X) \geq 3$ it is very hard to classify such varieties in terms of their birational invariants. Surprisingly, when $X$ has many holomorphic 1-forms, it is sometimes possible to achieve classification results in any dimension. In [Ka], Kawamata showed that: If $X$ is a smooth complex projective variety with $\kappa(X) = 0$ then the Albanese morphism $a : X \longrightarrow \Lambda(X)$ is surjective. If moreover, $q(X) = \dim(X)$, then $X$ is birational to an abelian variety. Subsequently, Kollár proved an effective version of this result (cf. [Ko2]): If $X$ is a smooth complex projective variety with $P_m(X) = 1$ for some $m \geq 4$, then the Albanese morphism $a : X \longrightarrow \Lambda(X)$ is surjective. If moreover, $q(X) = \dim(X)$, then $X$ is birational to an abelian variety. These results where further refined and expanded as follows:

Theorem 1.1 (T1). (cf. [CH1], [CH3], [HP], [Hac2]) If $P_m(X) = 1$ for some $m \geq 2$ or if $P_3(X) \leq 3$, then the Albanese morphism $a : X \longrightarrow \Lambda(X)$ is surjective. If moreover $q(X) = \dim(X)$, then:

1. If $P_m(X) = 1$ for some $m \geq 2$, then $X$ is birational to an abelian variety.

2. If $P_3(X) = 2$, then $\kappa(X) = 1$ and $X$ is a double cover of its Albanese variety.

3. If $P_3(X) = 3$, then $\kappa(X) = 1$ and $X$ is a bi-double cover of its Albanese variety.

In this paper we will prove a similar result for varieties with $P_3(X) = 4$ and $q(X) = \dim(X)$. We start by considering the following examples:

Example 1. Let $G$ be a group acting faithfully on a curve $C$ and acting faithfully by translations on an abelian variety $\tilde{K}$, so that $C/G = E$ is an elliptic curve and $\dim H^0(C, \omega_C \otimes 3) = 4$. Let $G$ act diagonally on $\tilde{K} \times C$, then $X := \tilde{K} \times C/G$ is a smooth projective variety with $\kappa(X) = 1$, $P_3(X) = 4$ and $q(X) = \dim(X)$. We illustrate some examples below:

1. $G = \mathbb{Z}_m$ with $m \geq 3$. Consider an elliptic curve $E$ with a line bundle $L$ of degree 1. Taking the normalization of the $m$-th root of a divisor $B = (m-a)B_1 + aB_2 \in mL$ with $1 \leq a \leq m-1$ and $m \geq 3$, one obtains a smooth curve $C$ and a morphism $g : C \longrightarrow E$ of degree $m$. One has that

$$g_* \omega_C = \sum_{i=0}^{m-1} L^{(i)}$$

where $L^{(i)} = L \otimes (-\frac{iB}{m})$ for $i = 0, ..., m-1$.

2. $G = \mathbb{Z}_2$. Let $L$ be a line bundle of degree 2 over an elliptic curve $E$. Let $C \longrightarrow E$ be the degree 2 cover defined by a reduced divisor $B \in |2L|$. 2000 MSC. 14J10
If \( \omega \) are ramified along 4 points and hence one has double covers \( P \) morphisms, \( \dim \) degree 2 cover \( a : \) generate a subgroup of \( \text{Pic} \) variety, we let additive and multiplicative notation interchangeably. If we work over the field of complex numbers. We \( \text{Notation and conventions.} \)

Grant no: MDA904-03-1-0101 and by a grant from the Sloan Foundation.

The second author was partially supported by NSA research.

\( G = (\mathbb{Z}_2)^2 \). Let \( L_i \) for \( i = 1, 2 \) be line bundles of degree 1 on an elliptic curve \( E \) and \( C_i \to E \) be degree 2 covers defined by disjoint reduced divisors \( B_i \in |2L_i| \). Then \( C := C_1 \times_E C_2 \to E \) is a \( G \) cover.

\( G = (\mathbb{Z}_2)^3 \). For \( i = 1, 2, 3, 4 \), let \( P_i \) be distinct points on an elliptic curve \( E \). For \( j = 1, 2, 3 \) let \( L_j \) be line bundles of degree 1 on \( E \) such that \( B_j = P_1 + P_2 \in |2L_1|, B_2 = P_1 + P_3 \in |2L_2| \) and \( B_3 = P_1 + P_4 \in |2L_3| \). Let \( C_j \to E \) be degree 2 covers defined by reduced divisors \( B_j \in |2L_j| \). Let \( C \) be the normalization of \( C_1 \times_E C_2 \times_E C_3 \to E \), then \( C \) is a \( G \) cover.

Note that (1) is ramified at 2 points. Following [Be] VI.12, one has that \( P_2(X) = \dim H^0(C, \omega_C^{\otimes 2})^G = 2 \) and \( P_3(X) = \dim H^0(C, \omega_C^{\otimes 3})^G = 4 \). Similarly (2), (3), (4) are ramified along 4 points and hence \( P_2(X) = P_3(X) = 4 \).

**Example 2.** Let \( q : A \to S \) be a surjective morphism with connected fibers from an abelian variety of dimension \( n \geq 3 \) to an abelian surface. Let \( L \) be an ample line bundle on \( S \) with \( h^0(S, L) = 1 \), \( P \in \text{Pic}^0(A) \) with \( P \notin \text{Pic}^0(S) \) and \( P \otimes L^2 \in \text{Pic}^0(S) \). For \( D \) an appropriate reduced divisor in \( |L \otimes P \otimes L^2| \), there is a degree 2 cover \( a : X \to A \) such that \( a_*(\mathcal{O}_X) = \mathcal{O}_A \otimes (L \otimes P)^\ell \). One sees that \( P_i(X) = 1, 4, 4 \) for \( i = 1, 2, 3 \).

**Example 3.** Let \( q : A \to E_1 \times E_2 \) be a surjective morphism from an abelian variety to the product of two elliptic curves, \( p_i : A \to E_i \) the corresponding morphisms, \( L_i \) be line bundles of degree 1 on \( E_i \) and \( P, Q \in \text{Pic}^0(A) \) such that \( P, Q \) generate a subgroup of \( \text{Pic}^0(A)/\text{Pic}^0(E_1 \times E_2) \) which is isomorphic to \( (\mathbb{Z}_2)^2 \). Then one has double covers \( X_i \to A \) corresponding to divisors \( D_i \in |2(q_1^*L_1 \otimes P), D_2 \in |2(q_2^*L_2 \otimes Q)| \). The corresponding bi-double cover satisfies

\[
\text{One sees that } P_i(X) = 1, 4, 4 \text{ for } i = 1, 2, 3.
\]

We will prove the following:

**Theorem 1.2 (T2).** Let \( X \) be a smooth complex projective variety with \( P_2(X) = 4 \), then the Albanese morphism \( a : X \to A \) is surjective (in particular \( q(X) \leq \dim(X) \)). If moreover, \( q(X) = \dim(X) \), then \( \kappa(X) \leq 2 \) and we have the following cases:

1. If \( \kappa(X) = 2 \), then \( X \) is birational either to a double cover or to a bi-double cover of \( A \) as in Examples 2 and 3 and so \( P_2(X) = 4 \).
2. If \( \kappa(X) = 1 \), then \( X \) is birational to the quotient \( \tilde{K} \times C/G \) where \( C \) is a curve, \( \tilde{K} \) is an abelian variety, \( G \) acts faithfully on \( C \) and \( \tilde{K} \). One has that either \( P_2(X) = 2 \) and \( C \to C/G \) is branched along 2 points with inertia group \( H \cong \mathbb{Z}_s \) with \( s \geq 3 \) or \( P_2(X) = 4 \) and \( C \to C/G \) is branched along 4 points with inertia group \( H \cong (\mathbb{Z}_2)^s \) with \( s \in \{1, 2, 3\} \). See Example 1.

**Acknowledgments.** The second author was partially supported by NSA research grant no: MDA904-03-1-0101 and by a grant from the Sloan Foundation.

**Notation and conventions.** We work over the field of complex numbers. We identify Cartier divisors and line bundles on a smooth variety, and we use the additive and multiplicative notation interchangeably. If \( X \) is a smooth projective variety, we let \( K_X \) be a canonical divisor, so that \( \omega_X = \mathcal{O}_X(K_X) \), and we denote by \( \kappa(X) \) the Kodaira dimension, by \( q(X) := h^1(\mathcal{O}_X) \) the irregularity and by \( P_m(X) := h^0(\omega_X^m) \) the \( m \)-th plurigenera. We denote by \( a : X \to A(X) \) the Albanese map and by \( \text{Pic}^0(X) \) the dual abelian variety to \( A(X) \) which parameterizes all topologically trivial line bundles on \( X \). For a \( \mathbb{Q} \)-divisor \( D \) we let \( [D] \) be the integral part and \{\( D \)\} the fractional part. Numerical equivalence is denoted by \( \equiv \) and we write
2. Preliminaries

2.1. The Albanese map and the Iitaka fibration. Let $X$ be a smooth projective variety. If $\kappa(X) > 0$, then the Iitaka fibration of $X$ is a morphism of projective varieties $f : X' \to Y$, with $X'$ birational to $X$ and $Y$ of dimension $\kappa(X)$, such that the general fiber of $f$ is smooth, irreducible, of Kodaira dimension zero. The Iitaka fibration is determined only up to birational equivalence. Since we are interested in questions of a birational nature, we usually assume that $X = X'$ and that $Y$ is smooth.

$X$ has maximal Albanese dimension if $\dim(a_X(X)) = \dim(X)$. We will need the following facts (cf. [HP] Propositions 2.1, 2.3, 2.12 and Lemma 2.14 respectively).

Proposition 2.1 (albanese). Let $X$ be a smooth projective variety of maximal Albanese dimension, and let $f : X \to Y$ be the Iitaka fibration (assume $Y$ smooth). Denote by $f_* : A(X) \to A(Y)$ the homomorphism induced by $f$ and consider the commutative diagram:

\[
\begin{array}{c}
X \\
\downarrow f \\
Y
\end{array} \quad \quad \quad \begin{array}{c}
\overset{a_X}{\longrightarrow} \quad A(X) \\
\downarrow f_* \\
\overset{a_Y}{\longrightarrow} \quad A(Y)
\end{array}
\]

Then:

a) $Y$ has maximal Albanese dimension;

b) $f_*$ is surjective and $\ker f_*$ is connected of dimension $\dim(X) - \kappa(X)$;

c) There exists an abelian variety $P$ isogenous to $\ker f_*$ such that the general fiber of $f$ is birational to $P$.

Let $K := \ker f_*$ and $F = F_{X/Y}$. Define

\[ G := \ker \left( \text{Pic}^0(X) \to \text{Pic}^0(F) \right). \]

Then

Lemma 2.2 (LG). $G$ is the union of finitely many translates of $\text{Pic}^0(Y)$ corresponding to the finite group

\[ \overline{G} := G/\text{Pic}^0(Y) \cong \ker \left( \text{Pic}^0(K) \to \text{Pic}^0(F) \right). \]

2.2. Sheaves on abelian varieties. Recall the following easy corollary of the theory of Fourier-Mukai transforms cf. [M]:

Proposition 2.3 (inclusion). Let $\psi : \mathcal{F} \hookrightarrow \mathcal{G}$ be an inclusion of coherent sheaves on an abelian variety $A$ inducing isomorphisms $H^i(A, \mathcal{F} \otimes P) \to H^i(A, \mathcal{G} \otimes P)$ for all $i \geq 0$ and all $P \in \text{Pic}^0(A)$. Then $\psi$ is an isomorphism of sheaves.

Following [M], we will say that a coherent sheaf $\mathcal{F}$ on an abelian variety $A$ is I.T. 0 if $h^i(A, \mathcal{F} \otimes P) = 0$ for all $i > 0$. We will say that an inclusion of coherent sheaves on $A$, $\psi : \mathcal{F} \hookrightarrow \mathcal{G}$ is an I.T. 0 isomorphism if $\mathcal{F}, \mathcal{G}$ are I.T. 0 and $h^0(\mathcal{G}) = h^0(\mathcal{F})$. From the above proposition, it follows that every I.T. 0 isomorphism $\mathcal{F} \hookrightarrow \mathcal{G}$ is an isomorphism. We will need the following result:
Lemma 2.4 (L1). Let $f : X \to E$ be a morphism from a smooth projective variety to an elliptic curve, such that $K_X$ is $E$-big. Then, for all $P \in \text{Pic}^0(X)_{\text{tors}}$, $\eta \in \text{Pic}^0(E)$ and all $m \geq 2$, $f_* (\omega_X^m \otimes P \otimes f^* \eta)$ is I.T. 0. In particular
\[ \deg(f_* (\omega_X^m \otimes P \otimes f^* \eta)) = h^0(\omega_X^m \otimes P \otimes f^* \eta). \]

The proof of the above lemma is analogous to the proof of Lemma 2.6 of [Hac2]. We just remark that it suffices to show that $f_* (\omega_X^m \otimes P)$ is I.T. 0. By [Ko1], one sees that $f_* (\omega_X^m \otimes P)$ is torsion free and hence locally free on $E$. By Riemann-Roch
\[ h^0(\omega_X^m \otimes P) = h^0(f_* (\omega_X^m \otimes P)) = \chi(f_* (\omega_X^m \otimes P)) = \deg(f_* (\omega_X^m \otimes P)). \]

2.3. Cohomological support loci. Let $\pi : X \to A$ be a morphism from a smooth projective variety to an abelian variety, $T \subset \text{Pic}^0(A)$ the translate of a subtorus and $\mathcal{F}$ a coherent sheaf on $X$. One can define the cohomological support loci of $\mathcal{F}$ as follows:
\[ V^i(X, T, \mathcal{F}) := \{ P \in T | h^i(X, \mathcal{F} \otimes \pi^* P) > 0 \}. \]
If $T = \text{Pic}^0(X)$ we write $V^i(\mathcal{F})$ or $V^i(X, \mathcal{F})$ instead of $V^i(X, \text{Pic}^0(X), \mathcal{F})$. When $\mathcal{F} = \omega_X$, the geometry of the loci $V^i(\omega_X)$ is governed by the following result of Green and Lazarsfeld (cf. [GL], [EL]):

**Theorem 2.5 (genvanish). (Generic Vanishing Theorem)** Let $X$ be a smooth projective variety. Then:

a) $V^i(\omega_X)$ has codimension $\geq i - (\dim(X) - \dim(a_X(X)))$;
b) Every irreducible component of $V^i(X, \omega_X)$ is a translate of a subtorus of $\text{Pic}^0(X)$ by a torsion point (the same also holds for the irreducible components of $V^i_m(\omega_X) := \{ P \in \text{Pic}^0(X) | h^i(X, \omega_X \otimes P) \geq m \}$);
c) Let $T$ be an irreducible component of $V^i(\omega_X)$, let $P \in T$ be a point such that $V^i(\omega_X)$ is smooth at $P$, and let $v \in H^1(X, \mathcal{O}_X) \cong T_P \text{Pic}^0(X)$. If $v$ is not tangent to $T$, then the sequence
\[ H^{i-1}(X, \omega_X \otimes P) \xrightarrow{\text{res}} H^i(X, \omega_X \otimes P) \xrightarrow{\text{res}} H^{i+1}(X, \omega_X \otimes P) \]
is exact. Moreover, if $P$ is a general point of $T$ and $v$ is tangent to $T$ then both maps vanish;
d) If $X$ has maximal Albanese dimension, then there are inclusions:
\[ V^0(\omega_X) \supseteq V^1(\omega_X) \supseteq \cdots \supseteq V^n(\omega_X) = \{ \mathcal{O}_X \}. \]
e) Let $f : Y \to X$ be a surjective map of projective varieties, $Y$ smooth, then statements analogous to a), b), c) for $P \in \text{Pic}^0_{\text{tors}}(Y)$ and d) above also hold for the sheaves $R^i f_* \omega_X$. More precisely we refer to [CH3], [CIH] and [Hac5].

When $X$ is of maximal Albanese dimension, its geometry is very closely connected to the properties of the loci $V^i(\omega_X)$. We recall the following two results from [CH2]:

**Theorem 2.6 (TCH2).** Let $X$ be a variety of maximal Albanese dimension. The translates through the origin of the irreducible components of $V^0(\omega_X)$ generate a subvariety of $\text{Pic}^0(X)$ of dimension $\kappa(X) - \dim(X) + q(X)$. In particular, if $X$ is of general type then $V^0(X, \omega_X)$ generates $\text{Pic}^0(X)$.

**Proposition 2.7 (PCH2).** Let $X$ be a variety of maximal Albanese dimension and $G, Y$ defined as in Proposition 2.1. Then

a) $V^0(X, \text{Pic}^0(X), \omega_X) \subset G$;
b) For every $P \in G$, the loci $V^0(X, \text{Pic}^0(X), \omega_X) \cap (P + \text{Pic}^0(Y))$ are non-empty;
c) If $P$ is an isolated point of $V^0(X, \text{Pic}^0(X), \omega_X)$, then $P = \mathcal{O}_X$. 
The following result governs the geometry of \( V^0(\omega_X^{\otimes m}) \) for all \( m \geq 2 \):

**Proposition 2.8 (Pm).** Let \( X \) be a smooth projective variety of maximal Albanese dimension, \( f : X \to Y \) the Iitaka fibration (assume \( Y \) smooth) and \( G \) defined as in Proposition 2.1. If \( m \geq 2 \), then \( V^0(\omega_X^{\otimes m}) = G \). Moreover, for any fixed \( Q \in V^0(\omega_X^{\otimes m}) \), and all \( P \in \text{Pic}^0(Y) \) one has \( h^0(\omega_X^{\otimes m} \otimes Q \otimes P) = h^0(\omega_X^{\otimes m} \otimes Q) \).

We will also need the following lemma proved in [CH2] §3.

**Lemma 2.9 (L7).** Let \( X \) be a smooth projective variety and \( E \) an effective \( a_X \)-exceptional divisor on \( X \). If \( \mathcal{O}_X(E) \otimes P \) is effective for some \( P \in \text{Pic}^0(X) \), then \( P = \mathcal{O}_X \).

The following result is due to Ein and Lazarsfeld (see [HP] Lemma 2.13):

**Lemma 2.10 (Lel).** Let \( X \) be a variety such that \( \chi(\omega_X) = 0 \) and such that \( a_X : X \to \text{A}(X) \) is surjective and generically finite. Let \( T \) be an irreducible component of \( V^0(\omega_X) \), and let \( \pi_E : X \to E : = \text{Pic}^0(T) \) be the morphism induced by the map \( \text{A}(X) \to \text{Pic}^0(\text{Pic}^0(X)) \to E \) corresponding to the inclusion \( T \to \text{Pic}^0(X) \).

Then there exists a divisor \( D_T \prec R := \text{Ram}(\omega_X) = K_X \), vertical with respect to \( \pi_E \) (i.e. \( \pi_E(D_T) \notin E \)), such that for general \( P \in T \), \( G_T := R - D_T \) is a fixed divisor of each of the linear series \( [K_X + P] \).

We have the following useful Corollary

**Corollary 2.11 (C9).** In the notation of Lemma 2.10, if \( \text{dim}(T) = 1 \), then for any \( P \in T \), there exists a line bundle of degree 1 on \( E \) such that \( \pi_E^*L_P \sim K_X + P \).

**Proof.** By [HP] Step 8 of the proof of Theorem 6.1, for general \( Q \in T \), there exists a line bundle of degree 1 on \( E \) such that \( \pi_E^*L_Q \sim K_X + Q \). Write \( P = Q + \pi^*\eta \) where \( \eta \in \text{Pic}^0(E) \). Then, since
\[ h^0(\omega_X \otimes P \otimes \pi^*(L_Q \otimes \eta)^\vee) = h^0(\pi^*(\omega_X \otimes Q) \otimes L_Q^\vee) \neq 0, \]
one sees that there is an inclusion \( \pi^*(L_Q \otimes \eta) \to \omega_X \otimes P \). \( \square \)

Recall the following result (cf. [Hac2] Lemma 2.17):

**Lemma 2.12 (claimA).** Let \( X \) be a smooth projective variety, let \( L \) and \( M \) be line bundles on \( X \), and let \( T \subset \text{Pic}^0(X) \) be an irreducible subvariety of dimension \( t \). If for all \( P \in T \), \( \text{dim} |L + P| \geq a \) and \( \text{dim} |M - P| \geq b \), then \( \text{dim} |L + M| \geq a + b + t \).

**Lemma 2.13 (fiber).** Let \( T \) be a 1-dimensional component of \( V^0(\omega_X) \), \( E := T^\vee \) and \( \pi : X \to E \) the induced morphism. Then \( P|F \cong \mathcal{O}_F \) for all \( P \in T \).

**Proof.** Let \( G_T, D_T \) be as in Lemma 2.10, then for \( P \in T \) we have \( |K_X + P| = G_T + |D_T + P| \) and hence the divisor \( D_T + P \) is effective. It follows that \( (D_T + P)|F \) is also effective. However \( D_T \) is vertical with respect to \( \pi \) and hence \( D_T|F \cong \mathcal{O}_F \). By Lemma 2.9, one sees that \( P|F \cong \mathcal{O}_F \). \( \square \)

3. **Kodaira dimension of Varieties with \( P_3(X) = 4 \), \( q(X) = \text{dim}(X) \)**

The purpose of this section is to study the Albanese map and Iitaka fibration of varieties with \( P_3 = 4 \) and \( q = \text{dim}(X) \). We will show that: 1) the Albanese map is surjective, 2) the image of the Iitaka fibration is an abelian variety (and hence the Iitaka fibration factors through the Albanese map), 3) we have that \( \kappa(X) \leq 2 \).

We begin by fixing some notation. We write
\[ V_0(X, \omega_X) = \bigcup_{i \in I} S_i \]
where \( S_i \) are irreducible components. Let \( T_i \) denote the translate of \( S_i \) passing through the origin and \( \delta_i := \text{dim}(S_i) \). In particular, \( S_0 \) denotes the component containing the origin. For any \( i, j \in I \), let \( \delta_{ij} := \text{dim}(T_i \cap T_j) \).
Recall that $V_\eta(X, \omega_X) \subset G \to \tilde{G} := G/	ext{Pic}(Y)$. For any $\eta \in \tilde{G}$, let $S_\eta$ denote a maximal dimensional component which maps to $\eta$. If $X$ is of maximal Albanese dimension with $q(X) = \dim(X)$, then its Iitaka fibration image $Y$ is of maximal Albanese dimension with $q(Y) = \dim(Y) = \kappa(X)$. Moreover, by Proposition 2.7, one has $\delta_i \geq 1, \forall i \neq 0$.

Now let $Q_i$ ($Q_{\eta}$ resp.) be a general torsion element in $S_i$ ($S_{\eta}$ resp.), we denote by $P_{m,i} := h^0(X, \omega_X^{\otimes m} \otimes Q_i)$ ($P_{m,\eta}$ resp.). Proposition 2.8 can be rephrased as

(1) $[pm] P_{m,\eta} = P_{m,\eta+\zeta} \quad \forall \eta \in \text{Pic}^0(X), \zeta \in \text{Pic}^0(Y), \ m \geq 2$.

By Lemma 2.12 one has, for any $\eta, \zeta \in G$,

(2) $[\text{plur}] \left\{ \begin{array}{ll} P_{2,\eta+\zeta} \geq P_{1,\eta} + P_{1,\zeta} + \delta_{\eta,\zeta} - 1, \\
P_{2,2\eta} \geq 2P_{1,\eta} + \delta_{\eta} - 1, \\
P_{2,\eta+\zeta} \geq P_{1,\eta} + P_{2,\zeta} + \delta_{\eta} - 1. \end{array} \right.$

The following lemma is very useful when $\kappa \geq 2$.

**Lemma 3.1 (elliptic).** Let $X$ be a variety of maximal Albanese dimension with $\kappa(X) \geq 2$. Suppose that there is a surjective morphism $\pi : X \to E$ to an elliptic curve $E$, and suppose that there is an inclusion $\varphi : \pi^* L \to \omega_X^{\otimes m} \otimes P$ for some $m \geq 2$, $P_{\eta} = \mathcal{O}_F$ where $F$ is a general fiber of $\pi$ and $L$ is an ample line bundle on $E$. Then the induced map $L \to \pi_* (\omega_X^{\otimes m} \otimes P)$ is not an isomorphism, $\text{rank}(\pi_* (\omega_X^{\otimes m} \otimes P)) \geq 2$ and $h^0(X, \omega_X^{\otimes m} \otimes P) > h^0(E, L)$.

**Proof.** By the easy addition theorem, $\kappa(F) \geq 1$. Hence by Theorem 1.1, $P_{m,F} \geq 2$ for $m \geq 2$. The sheaf $\pi_* (\omega_X^{\otimes m} \otimes P)$ has rank equal to $h^0(F, \omega_X^{\otimes m} \otimes P|_F) = h^0(F, \omega_F^{\otimes m}) \geq 2$. Therefore, $L \to \pi_* (\omega_X^{\otimes m} \otimes P)$ is not an isomorphism. Since they are non-isomorphic I.T.0 sheaves, it follows that $h^0(\pi_* (\omega_X^{\otimes m} \otimes P)) > h^0(L)$. □

**Corollary 3.2 (ell).** Keep the notation as in Lemma 3.1. If there is a morphism $\pi' : X \to E'$ and an inclusion $\pi'^* L' \hookrightarrow \omega_X \otimes P'$ for some ample $L'$ on $E'$ and $P \in \text{Pic}^0(X)$ with $P_{\eta,F} = \mathcal{O}_{F'}$ then for all $m \geq 2$

$P_{m+1}(X) \geq 2 + h^0(X, \omega_X^{\otimes m} \otimes P) > 2 + h^0(E', L')$.

**Proof.** The inclusion $\pi'^* L' \hookrightarrow \omega_X \otimes P'$ induces an inclusion

$\pi'^* L' \otimes \omega_X^{\otimes m} \otimes P \hookrightarrow \omega_X^{\otimes m+1}$.

By Riemann-Roch, one has

$P_{m+1}(X) \geq h^0(E', L' \otimes \pi'^* (\omega_X^{\otimes m} \otimes P)) \geq h^0(E', \pi'^* (\omega_X^{\otimes m} \otimes P) + \text{rank}(\pi'^* (\omega_X^{\otimes m} \otimes P))$.

Proposition 2.7, there exists $\eta \in \text{Pic}^0(Y)$ such that $h^0(\omega_X^{\otimes m-1} \otimes P^{\otimes 2} \otimes \eta) \neq 0$ and hence there is an inclusion

$\pi'^* L' \hookrightarrow \omega_X^{\otimes m} \otimes P \otimes \eta$.

By Proposition 2.8 and Lemma 3.1,

$h^0(X, \omega_X^{\otimes m} \otimes P) = h^0(X, \omega_X^{\otimes m} \otimes P \otimes \eta) > h^0(E', L')$.

□

**Remark 3.3 (1dim).** Let $X$ be a variety with $\kappa(X) \geq 2$. Suppose that there is a 1-dimensional component $S_i \subset V^0(\omega_X)$. We often consider the induced map $\pi : X \to E := T_i^\vee$. It is easy to see that $\pi$ factors through the Iitaka fibration. By Corollary 2.11 and Lemma 2.13, there is an inclusion $\varphi : \pi^* L \to \omega_X \otimes P$ for some $P \in \text{Pic}^0(X)$ with $P_{\eta,F} = \mathcal{O}_F$ and some ample line bundle $L$ on $E$. In what follows, we will often apply Lemma 3.1 and Corollary 3.2 to this situation.
Lemma 3.4 (P2). Let $X$ be a variety of maximal Albanese dimension with $\kappa(X) \geq 2$ and $P_3(X) = 4$. Then for any $\zeta \in G - \text{Pic}^0(Y)$, one has $P_{2,\zeta} \leq 2$.

*Proof.* If $P_{2,\zeta} \geq 3$, then by (2) and Proposition 2.7, one sees that $V^0(\omega_X) \cap (\text{Pic}^0(Y) - \zeta)$ consists of 1-dimensional components. Let $S$ be one such component and $\pi : X \to E := S'\backslash$ be the induced morphism. Then there is an ample line bundle $L$ on the elliptic curve $E$ and an inclusion $L \hookrightarrow \pi_* (\omega_X \otimes Q)$ for some $Q \in \text{Pic}^0(Y) - \zeta$. By Corollary 3.2, $P_3(X) \geq 2 + P_{2,\zeta} \geq 5$ which is impossible. □

Theorem 3.5 (surj). Let $X$ be a smooth projective variety with $P_3(X) = 4$, then the Albanese morphism $a : X \to \Lambda$ is surjective.

*Proof.* We follow the proof of Theorem 5.1 of [HP]. Assume that $a : X \to \Lambda$ is not surjective, then we may assume that there is a morphism $f : X \to Z$ where $Z$ is a smooth variety of general type, of dimension at least 1, such that its Albanese map $a_Z : Z \to S$ is birational onto its image. By the proof of Theorem 5.1 of [HP], it suffices to consider the cases in which $P_1(Z) \leq 3$ and hence $\dim(Z) \leq 2$. If $\dim(Z) = 2$, then $q(Z) = \dim(S) \geq 3$ and since $\chi(\omega_Z) > 0$, one sees that $V^0(\omega_Z) = \text{Pic}^0(S)$. By the proof of Theorem 5.1 of [HP], one has that for generic $P \in \text{Pic}^0(S)$,

$$P_3(X) \geq h^0(\omega_Z \otimes P) + h^0((\omega_X \otimes f^* \omega_Y) \otimes P) + \dim(S) - 1 \geq 1 + 2 + 3 - 1 \geq 5.$$

This is a contradiction, so we may assume that $\dim(Z) = 1$. It follows that $q(Z) = q(Z) = P_1(Z) \geq 2$ and one may write $\omega_Z = L \otimes 2$ for some ample line bundle $L$ on $Z$.

Therefore, for general $P \in \text{Pic}^0(Z)$, one has $h^0(\omega_Z \otimes L \otimes P) \geq 2$ and proceeding as in the proof of Theorem 5.1 of [HP], that $h^0(\omega_X \otimes f^* (\omega_Z \otimes L)^{\vee} \otimes P) \geq 2$. It follows as above that

$$P_3(X) \geq h^0(\omega_Z \otimes L \otimes P) + h^0((\omega_X \otimes f^* (\omega_Z \otimes L)^{\vee} \otimes P) + \dim(S) - 1 \geq 2 + 2 + 2 - 1 \geq 5.$$

This is a contradiction and so $a : X \to \Lambda$ is surjective. □

Proposition 3.6 (gt). Let $X$ be a smooth projective variety with $P_3(X) = 4$, $q(X) = \dim(X)$, then

(1) $X$ is not of general type and

(2) if $\kappa(X) \geq 2$, then

$$V^0(\omega_X) \cap f^* \text{Pic}^0(Y) = \{ \mathcal{O}_X \}.$$

*Proof.* If $\kappa(X) = 1$, then clearly $X$ is not of general type as otherwise $X$ is a curve with $P_3(X) = 5q - 5 > 4$. We thus assume that $\kappa(X) \geq 2$. It suffices to prove (2) as then (1) will follow from Theorem 2.6.

If all points of $V^0(\omega_X) \cap f^* \text{Pic}^0(Y)$ are isolated, then the above statement follows from Proposition 2.7. Therefore, it suffices to prove that $\delta_0 = 0$. (Recall that $\delta_0$ is the maximal dimension of a component in $\text{Pic}^0(Y)$.)

Suppose that $\delta_0 \geq 2$. Then by (2) and Proposition 2.8, one has

$$P_2 \geq 1 + 1 + \delta_0 - 1 \geq 3, \quad P_3 \geq 3 + 1 + \delta_0 - 1 \geq 5$$

which is impossible.

Suppose now that $\delta_0 = 1$, i.e. there is a 1-dimensional component $T \subset V^0(\omega_X) \cap f^* \text{Pic}^0(Y)$. Let $\pi : X \to E := T'\backslash$ be the induced morphism. By Corollary 2.11, for some general $P \in T$, there exists a line bundle of degree 1 on $E$ and an inclusion $\pi^* L \hookrightarrow \omega_X \otimes P$. By Lemma 2.13, $P|F_{X/E} \equiv \mathcal{O}_{F_{X/E}}$.

We consider the inclusion $\varphi : L \otimes 2 \hookrightarrow \pi_* (\omega_X \otimes P \otimes 2)$. By Lemma 3.1, one sees that $h^0(\omega_X \otimes P \otimes 2) \geq 3$, and $\text{rank}(\pi_*(\omega_X \otimes P \otimes 2)) \geq 2$. So

$$P_3(X) = h^0(\omega_X \otimes P \otimes 3) \geq h^0(\omega_X \otimes P \otimes \pi^* L) =$$
Claim 3.10 \( \text{the inclusion} \) and this is the required contradiction. \( \square \)

Proposition 3.7 (Iitaka). Let \( X \) be a smooth projective variety with \( P_3(X) = 4 \), \( q(X) = \dim(X) \), and \( f : X \to Y \) be a birational model of its Iitaka fibration. Then \( Y \) is birational to an abelian variety.

Proof. Since \( X, Y \) are of maximal Albanese dimension, \( K_{X/Y} \) is effective. If \( h^0(\omega_Y \otimes P) > 0 \), it follows that \( h^0(\omega_X \otimes f^*P) > 0 \) and so by Proposition 3.6, \( f^*P = O_X \). By Proposition 2.1, the map \( f^* : \text{Pic}^0(Y) \to \text{Pic}^0(X) \) is injective and hence \( P = O_Y \). Therefore \( V^0(\omega_Y) = \{O_Y\} \) and by Theorem 2.6, one has \( \kappa(Y) = 0 \) and hence \( Y \) is birational to an abelian variety. \( \square \)

We are now ready to describe the cohomological support loci of varieties with \( \kappa(X) \geq 2 \) explicitly. Recall that by Proposition 2.7, for all \( \eta \neq 0 \in \bar{G}, \delta_\eta \geq 1 \).

Theorem 3.8. Let \( X \) be a smooth projective variety with \( P_3(X) = 4, q(X) = \dim(X) \) and \( \kappa(X) \geq 2 \). Then \( \kappa(X) = 2 \) and \( \bar{G} \cong (\mathbb{Z}_2)^s \) for some \( s \geq 1 \).

Proof. The proof consists of following claims.

Claim 3.9 (c11). If \( \kappa(X) \geq 2 \) and \( T \subset V^0(\omega_X) \) is a positive dimensional component, then \( T + T \subset \text{Pic}^0(Y) \), i.e. \( \bar{G} \cong (\mathbb{Z}_2)^s \).

Proof of Claim 3.9. It suffices to prove that \( 2\eta = 0 \) for \( 0 \neq \eta \in \bar{G} \). Suppose that \( 2\eta \neq 0 \), we will find a contradiction.

We first consider the case that \( \delta_\eta \geq 2 \) and \( \delta_{-2\eta} \geq 2 \). Then by (2), \( P_{2,2\eta} \geq 1 + 1 + \delta_\eta - 1 \geq 3, P_5 \geq 3 + 1 + \delta_{-2\eta} - 1 \geq 5 \) which is impossible.

We then consider the case that \( \delta_\eta \geq 2 \) and \( \delta_{-2\eta} = 1 \). Again we have \( P_{2,2\eta} \geq 3 \).

We consider the induced map \( \pi : X \to E := T_{-2\eta}^\vee \) and the inclusion \( \varphi : \pi^*L \to \omega_X \otimes Q_{-\eta} \) where \( E \) is an elliptic curve and \( L \) is an ample line bundle on \( E \). It follows that there is an inclusion

\[ \pi^*L \otimes (\omega_X \otimes Q_\eta)^{\otimes 2} \to \omega_X^{3} \otimes Q_\eta^{\otimes 2} \otimes Q_{-\eta}. \]

By Lemma 3.1, one has that \( \text{rank}(\pi_*(\omega_X \otimes Q_\eta)^{\otimes 2}) \geq 2 \). By Proposition 2.8, Riemann-Roch and Lemma 2.4

\[ P_3(X) = h^0(\omega_X^{3} \otimes Q_\eta^{\otimes 2} \otimes Q_{-\eta}) \geq h^0(\pi^*L \otimes (\omega_X \otimes Q_\eta)^{\otimes 2}) = \]

\[ h^0((\omega_X \otimes Q_\eta)^{\otimes 2}) + \text{rank}(\pi_*(\omega_X \otimes Q_\eta)^{\otimes 2}) \geq P_{2,2\eta} + 2 \geq 5, \]

which is impossible.

Lastly, we consider the case that \( \delta_\eta = 1 \). There is an induced map \( \pi : X \to E := T_{-\eta}^\vee \) and an inclusion \( \pi^*L \to \omega_X \otimes Q_\eta \). Hence there is an inclusion \( \varphi : \pi^*L^{\otimes 2} \to (\omega_X \otimes Q_\eta)^{\otimes 2} \). By Lemma 3.1, we have \( P_{2,2\eta} \geq 3 \). We now proceed as in the previous cases.

Therefore, any element \( \eta \in \bar{G} \) is of order 2 and hence \( \bar{G} \cong (\mathbb{Z}_2)^s \). \( \square \)

Claim 3.10 (c1fin). If there is a surjective map with connected fibers to an elliptic curve \( \pi : X \to E \) and an inclusion \( \pi^*L \to \omega_X \otimes P \) for an ample line bundle \( L \) on \( E \) and \( P \in \text{Pic}^0(X) \) (in particular if \( \delta_i = 1 \) for some \( i \neq 0 \) cf. Corollary 2.11). Then \( \kappa(X) = 2 \).

Proof of Claim 3.10. Since \( K_X \) is effective, there is also an inclusion \( L \to \pi_*(\omega_X^{\otimes 2} \otimes P) \).

By Lemma 3.1, one has \( \text{rank}(\pi_*(\omega_X^{\otimes 2} \otimes P)) \geq 2 \), \( h^0(\pi_*(\omega_X^{\otimes 2} \otimes P)) \geq 2 \). Consider the inclusion

\[ \pi_*(\omega_X^{\otimes 2} \otimes P) \otimes L \to \pi_*(\omega_X^{\otimes 3} \otimes P^{\otimes 2}). \]

Since

\[ P_3(X) = h^0(\pi_*(\omega_X^{\otimes 3} \otimes P^{\otimes 2})) \geq h^0(\pi_*(\omega_X^{\otimes 2} \otimes P) \otimes L) \geq \]

\[ \geq \text{rank}(\pi_*(\omega_X^{\otimes 2} \otimes P)) \geq 2. \]
For all \( i \neq 0, P_{1,i} = 1. \)

Claim 3.11. For all \( i \neq 0, P_{1,i} = 1. \)

Proof of the Claim 3.11. If \( P_{1,i} \geq 2, \) then by (2),
\[
4 \geq P_2 \geq 2P_{1,i} + \delta_i - 1.
\]
It follows that \( \delta_i = 1. \) Let \( E = T^\vee \) and \( \pi : X \to E \) be the induced morphism. One has an inclusion \( \pi^*L \to \omega_X \otimes Q_i. \) By Lemma 2.10, one has \( h^0(E, L) = h^0(\omega_X \otimes Q_i) \geq 2. \) Consider the inclusion \( \pi^*L \otimes Q_i \to \omega_X \otimes Q_i^2. \) By Lemma 3.1, one sees that
\[
P_3 \geq P_{2,2i} = h^0(\omega_X \otimes Q_i \otimes \omega_X) > h^0(E, L \otimes 2) \geq 4,
\]
which is impossible. \( \Box \)

Claim 3.12. If \( \kappa(X) = \dim(S) \) for some component \( S \) of \( V^0(\omega_X), \) then \( \kappa(X) = 2. \)

Proof of Claim 3.12. Let \( Q \) be a general point in \( S, \) and \( T \) be the translate of \( S \) through the origin. By Proposition 3.7, one sees that the induced map \( X \to T^\vee \) is isomorphic to the Itaka fibration. We therefore identify \( Y \) with \( T^\vee. \) We assume that \( \dim(S) \geq 3 \) and derive a contradiction. First of all, by (2)
\[
P_3(X) = h^0(\omega_X \otimes Q^2) \geq h^0(\omega_X \otimes Q) + \dim(S)
\]
and so \( h^0(\omega_X \otimes Q) = 1 \) and \( \dim(S) = 3. \)

Let \( H \) be an ample line bundle on \( Y \) and for \( m \) a sufficiently big and divisible integer, fix a divisor \( B \in [mK_X - f^*H]. \) After replacing \( X \) by an appropriate birational model, we may assume that \( B \) has simple normal crossings support. Let \( L = \omega_X \otimes O_X(-[B/m]), \) then \( L \equiv f^*(H + m) + \{B/m\} \) i.e. \( L \) is numerically equivalent to the sum of the pull back of an ample divisor and a k.l.t. divisor and so one has
\[
h^i(Y, f_*((\omega_X \otimes L \otimes Q) \otimes \eta)) = 0 \quad \text{for all } i > 0 \text{ and } \eta \in \text{Pic}^0(Y).
\]
Comparing the base loci, one can see that \( h^0(\omega_X \otimes L \otimes Q) = h^0(\omega_X \otimes Q) = 1 \) (cf. [CH1] Lemma 2.1 and Proposition 2.8) and so
\[
h^0(Y, f_*((\omega_X \otimes L \otimes Q) \otimes \eta)) = h^0(f_*((\omega_X \otimes L \otimes Q)) = 1 \quad \forall \eta \in \text{Pic}^0(Y).
\]
Since \( f_*((\omega_X \otimes L \otimes Q)) \) is a torsion free sheaf of generic rank one, by [Hac1] it is a principal polarization \( M. \)

Since one may arrange that \( [\frac{B}{m}] \prec K_X. \) There is an inclusion \( \omega_X \otimes Q \hookrightarrow \omega_X \otimes L \otimes Q. \) Pushing forward to \( Y, \) it induces an inclusion
\[
\varphi : f_*((\omega_X \otimes Q) \hookrightarrow M.
\]
Since \( f_*((\omega_X \otimes Q) \) is torsion free, it is generically of rank one. Hence it is of the form \( M \otimes I_Z \) for some ideal sheaf \( I_Z. \) However, \( h^0(Y, f_*((\omega_X \otimes Q) \otimes P) = h^0(M \otimes P \otimes I_Z) >
\]

for all $P \in \Pic^0(Y)$ and $M$ is a principal polarization. It follows that $\mathcal{I}_Z = \mathcal{O}_Y$ and thus $f_*(\omega_X \otimes Q) = M$. Therefore, one has an inclusion
\[ f^* M^\otimes 2 \hookrightarrow (\omega_X \otimes Q) \otimes (\omega_X \otimes L \otimes Q) \hookrightarrow \omega_X^\otimes 3 \otimes Q^\otimes 2. \]
It follows that
\[ 4 = P_3(X) = h^0(X, \omega_X^\otimes 3 \otimes Q^\otimes 2) \geq h^0(Y, M^\otimes 2) \geq 2^{\dim(S)}. \]
This is the required contradiction. □

**Claim 3.13 (cc4).** Any two components of $V^0(\omega_X)$ of dimension at least 2 must be parallel.

**Proof of Claim 3.13.** For $i = 1, 2$, let $S_i := T_i^\vee$ and $p_i : X \to S_i$ be the induced morphism. Assume that $\delta_1, \delta_2 \geq 2$ and $T_1, T_2$ are not parallel. By Lemma 2.10, one may write $K_X = G_i + D_i$ where $D_i$ is vertical with respect to $p_i : X \to S_i$ and for general $P \in T_i$, one has $|K_X + P| = G_i + |D_i + P|$ is a 0-dimensional linear system (see Claim 3.11).

Recall that we may assume that the image of the Iitaka fibration $f : X \to Y$ is an abelian variety. Pick $H$ an ample divisor on $Y$ and for $m$ sufficiently big and divisible integer, let
\[ B \in |mK_X - f^* H|. \]
After replacing $X$ by an appropriate birational model, we may assume that $B$ has normal crossings support. Let
\[ L := \omega_X(-\lfloor \frac{B}{m} \rfloor) \equiv \{ \frac{B}{m} \} + f^* H. \]
It follows that
\[ h^i(f_*(\omega_X \otimes L \otimes P) \otimes \eta) = 0 \quad \text{for all } i > 0, \eta \in \Pic^0(Y), \quad P \in \Pic^0(X). \]
The quantity $h^0(\omega_X \otimes L \otimes P \otimes f^* \eta)$ is independent of $\eta \in \Pic^0(Y)$. For some fixed $P \in T_1$ as above, and $\eta \in \Pic^0(S_1)$, one has a morphism
\[ |D_1 + P + \eta| \times |D_1 + P - \eta| \longrightarrow |2D_1 + 2P| \]
and hence $h^0(\omega_X(2D_1)^\otimes \otimes P^\otimes 2) \geq 3$. Similarly for some fixed $Q \in T_2$, and $\eta' \in \Pic^0(S_2)$, one has a morphism
\[ |D_2 + Q + \eta'| \times |K_X + L - Q + 2P - \eta'| \longrightarrow |K_X + L + D_2 + 2P| \]
and hence $h^0(\omega_X(D_2)^\otimes \otimes L \otimes P^\otimes 2) \geq 3$. It follows that since $h^0(\omega_X^\otimes 3 \otimes P^\otimes 2) = 4$, there is a 1 dimensional intersection between the images of the 2 morphisms above which are contained in the loci
\[ |2D_1 + 2P| + 2G_1 + K_X, \quad |K_X + L + D_2 + 2P| + |\frac{B}{m}| + G_2. \]
It is easy to see that for all but finitely many $P \in \Pic^0(X)$, one has $h^0(\omega_X \otimes P) \leq 1$. So there is a 1 parameter family $\tau_2 \subset \Pic^0(S_2)$ such that for $\eta' \in \tau_2$, one has that the divisor $D_{Q+\eta'} = |D_2 + Q + \eta'|$ is contained in $D_{P+\eta} + D_{P-\eta} + 2G_1 + K_X$ where $\eta \in \tau_1$ a 1 parameter family in $\Pic^0(S_1)$. Let $D_{Q+\eta'}$ be the components of $D_{Q+\eta'}$. which are not fixed for general $\eta' \in \tau_2$, then $D_{Q+\eta'}$ is not contained in the fixed divisor $2G_1 + K_X$ and hence is contained in some divisor of the form $D_{P+\eta} + D_{P-\eta}$ and hence is $S_1$ vertical.

If $\Pic^0(S_1) \cap \Pic^0(S_2) = \{ O_X \}$, then $D_{Q+\eta'}$ is a-exceptional, and this is impossible by Lemma 2.9.

If there is a 1-dimensional component $\Gamma \subset \Pic^0(S_1) \cap \Pic^0(S_2)$. Let $E = \Gamma^\vee$ and $\pi : X \to E$ be the induced morphism. The divisors $D_{Q+\eta'}$ are $E$-vertical. We may assume that $\pi$ has connected fibers. Since the $D_{Q+\eta'}$ vary with $\eta' \in \tau_2$, for general $\eta' \in \tau_2$, they contain a smooth fiber of $\pi$. So for general $\eta' \in \tau_2$ there is an
One has that

By the proof of [CH3] Theorem 4, one sees that

Claim 4.2

Claim 4.3

̸

inclusion π∗M → ωX⊗Q⊗π∗η′ where M is a line bundle of degree at least 1. By Claim 3.10, one has κ(X) = 2 and hence T1, T2 are parallel.

If there is a 2-dimensional component Γ ⊂ Pic0(S1)∩Pic0(S2), then δ1 = δ2 ≥ 3. By (2), one sees that P2,Q1+Q2 ≥ 3. By Lemma 3.4, this is impossible. □

By Claim 3.10, if there is a one dimensional component, then κ(X) = 2. Therefore, we may assume that δi ≥ 2 for all i ≠ 0. By Claim 3.13, since δi ≥ 2 for all i ≠ 0, then S1, Sj are parallel for all i, j ≠ 0. By Theorem 2.6, for an appropriate i ≠ 0, κ(X) = dim(S1) and so by Claim 3.12, one has κ(X) = 2. □

4. VARIETIES OF P3(X) = 4, q(X) = dim(X) AND k(X) = 2

In this section, we classify varieties with P3(X) = 4, q(X) = dim(X) and κ(X) = 2. The first first step is to describe the cohomological support loci of these varieties.

We must show that the only possible cases are the following (which corresponds to Examples 2 and 3 respectively):

1. G ≃ Z2, ν0(X,ωX) = {OX}∪Sn, δn = 2.
2. G ≃ Z2, ν0(X,ωX) = {OX}∪Sn∪Sn+ζ, δn = δζ = 1, δn+ζ = 2.

Using this information, we will determine the sheaves a∗(ωX) and this will enable us to prove the following:

Theorem 4.1 (Tk2). Let X be a smooth projective variety with P3(X) = 4, q(X) = dim(X) and κ(X) = 2, then X is one of the varieties described in Examples 2 and 3.

Proof. Recall that f : X → Y is a morphism birational to the Itaka fibration, Y is an abelian surface and f = q o a where q : A → Y.

Claim 4.2 (cl5). One has that f∗ωX = OY.

Proof of Claim 4.2. By Proposition 3.6, one has that V0(ωX)∩f∗Pic0(Y) = {OX}. By the proof of [CH3] Theorem 4, one sees that f∗ωX ≃ OY⊗H0(ωX). Since h0(ωX|FY) = 1, it follows that rank(f∗ωX) = 1 and hence f∗ωX ≃ OY. □

Claim 4.3 (cl6). Let T1, T2 be distinct components of V0(ωX) such that T1∩T2 ≠ 0, then T1∩T2 = P and

f∗(ωX⊗P) = L1⊗L2⊗Ip

where Y = E1×E2 and L1 are line bundles of degree 1 on the elliptic curves Ei and p is a point of Y.

Proof of Claim 4.3. Assume that P ∈ T1∩T2. Since κ(X) = 2, by Proposition 2.7, the Ti are 1-dimensional. Let πi : X → Ei := Ti be the induced morphisms. There are line bundles of degree 1, Li on Ei and inclusions πi∗Li → ωX⊗P (cf. Corollary 2.11).

We claim that rank(π1∗(ωX⊗P)) = 1. If this were not the case, then by Lemma 2.13

P1(FX/Ei) = rank(π1∗(ωX⊗P)) ≥ 2, P2(FX/Ei) = rank(π1∗(ωX⊗P)) ≥ 3 and so

P3(X) = h0(ωX⊗P⊗π1∗Li) = h0(π1∗(ωX⊗P)⊗L1) ≥ rank(π1∗(ωX⊗P)) + deg(π1∗(ωX⊗P))

and therefore

rank(π1∗(ωX⊗P)) = 3, deg(π1∗(ωX⊗P)) = 1.

Since rank(π1∗(ωX)) = rank(π1∗(ωX⊗P)), one has

deg(π1∗(ωX⊗P)) ≥ deg(π1∗(ωX)⊗L1) ≥ rank(π1∗(ωX)) ≥ 2,
which is impossible. Therefore, we may assume that
\[ \text{rank}(\pi_i \ast (\omega_X \otimes P)) = 1 \quad \text{for } i = 1, 2. \]
For any \( P_i \in T_i \), one has that \( P_i \otimes P_i' = \pi_i \ast \eta_i \) with \( \eta_i \in \text{Pic}^0(E_i) \). One sees that
\[ h^0(\omega_X \otimes P) = h^0(\pi_i \ast (\omega_X \otimes P) \otimes \eta_i) = h^0(\pi_i \ast (\omega_X \otimes P)) = h^0(\omega_X \otimes P). \]
If \( h^0(\omega_X \otimes P) \geq 2 \), then we may assume that \( L_1 := \pi_1 \ast (\omega_X \otimes P) \) is an ample line bundle of degree at least 2. From the inclusion \( \phi : L_1^{\otimes 2} \to \pi_1 \ast (\omega_X^{\otimes 2} \otimes P^{\otimes 2}) \), one sees that \( h^0(\omega_X^{\otimes 2} \otimes P^{\otimes 2}) = 4 \) and \( \phi \) is an I.T. 0 isomorphism (cf. Lemma 2.4) and so
\[ P_2(F_X/E_i) = h^0(\omega_X^{\otimes 2} \otimes P^{\otimes 2} \otimes I) = 1. \]
By Theorem 1.1, \( \kappa(F_X/E_i) = 0 \) and hence by easy addition, \( \kappa(X) \leq 1 \) which is impossible. Therefore we may assume that \( h^0(\omega_X \otimes P) = 1 \).

The coherent sheaf \( f_\ast (\omega_X \otimes P) \) is torsion free of generic rank 1 on \( Y \) and hence is isomorphic to \( L \otimes 2 \) and \( \mathcal{I} \) is an ideal sheaf cosped on at finitely many points. Let \( q_i : Y \to E_i \), so that \( \pi_i = q_i \circ f \). Since
\[ 1 = \text{rank}(\pi_1 \ast (\omega_X \otimes P)) = \text{rank}(q_{1i} \ast (L \otimes 2)) = \text{rank}(q_{1i} \ast L), \]
ones sees that \( L, F_{Y/E_i} \) is 1 and it easily follows that \( L = L_1 \otimes L_2 \) where \( L_i = q_{1i} \ast (L) \) is a line bundle of degree 1 on \( E_i \). Clearly, \( \mathcal{I} \) is the ideal sheaf of a point. \( \square \)

We will now consider the case in which \( \tilde{G} = Z_2 \). Let \( B \) be the branch locus of \( a : X \to A \). The divisor \( B \) is vertical with respect to \( q : A \to Y \) and hence we may write \( B = q^\ast \tilde{B} \). A normal variety and \( g \) is finite of degree 2 and so \( g_\ast O_Z = \mathcal{O}_A \otimes M' \) where \( M \) is a line bundle and the branch locus \( B \) is a divisor in \( |2M| \). The map \( F_{Z/Y} \to F_{A/Y} \) is étale of degree 2 and so \( M = q^\ast L \otimes I \) where \( P \) is a 2-torsion element of \( \text{Pic}^0(X) \). Let \( \nu : A' \to A \) be a birational morphism so that \( \nu^\ast B \) is a divisor with simple normal crossings support. Let \( B' = \nu^\ast B - 2\lfloor \nu^\ast (B/2) \rfloor \) and \( M' = \nu^\ast M(-\lfloor \nu^\ast (B/2) \rfloor) \). Let \( Z' \) be the normalization of \( Z \times_A A' \), and \( g' : Z' \to A \) be the induced morphism. Then \( g' \) is finite of degree 2, \( Z' \) is normal with rational singularities and \( g'_\ast (O_{Z'}) = \mathcal{O}_{A'} \otimes (M') \). Let \( \tilde{X} \) be an appropriate birational model of \( X \) such that there are morphisms \( \alpha : \tilde{X} \to A' \), \( v : \tilde{X} \to X \), \( \tilde{a} : \tilde{X} \to A \) and \( \beta : \tilde{X} \to Z' \). For all \( n \geq 0 \), one has that \( \beta_\ast (\omega_{\tilde{X}}^{\otimes n}) \cong \omega_{Z'}^{\otimes n} \). It follows that
\[ \alpha_\ast (\omega_{\tilde{X}}^{\otimes m}) = \omega_{\tilde{A'}} \otimes (M'^{\otimes m} \otimes \mathcal{I}). \]
Therefore
\[ \alpha_\ast (\omega_X) = \tilde{a}_\ast (\omega_{\tilde{X}}) = v_\ast \omega_{\tilde{A'}} \otimes \omega_{\tilde{A'}} \otimes M' = \mathcal{O}_A \otimes v_\ast (\omega_{\tilde{A'}} \otimes v_\ast (q^\ast L(-\lfloor \nu^\ast (B/2) \rfloor))) = \mathcal{O}_A \otimes q^\ast L \otimes I \otimes (\frac{B}{2}). \]

**Claim 4.4 (c19) If \( \tilde{G} = Z_2 \), then for any \( P \in V^0(\omega_X) \), one has**
\[ f_\ast (\omega_X \otimes P) \neq L_1 \otimes L_3 \otimes \mathcal{I}_p \]
where \( Y = E_1 \times E_2 \) and \( L_i \) are ample line bundles of degree 1 on \( E_i \) and \( p \) is a point of \( Y \).

**Proof of Claim 4.4.** If \( f_\ast (\omega_X \otimes P) = L_1 \otimes L_3 \otimes \mathcal{I}_p \), then \( B/2 \) is not log terminal. By [Hac3] Theorem 1, one sees that since \( B/2 \) is not log terminal, one has that \( |B/2| \neq 0 \) and this is impossible as then \( Z \) is not normal. \( \square \)

Combining Claim 4.3 and Claim 4.4, one sees that if \( \tilde{G} = Z_2 \), then \( V_0(X, \omega_X) = \{ \mathcal{O}_X \} \cup \mathcal{S}_0 \) with \( \delta_0 = 2 \). We then have the following:

**Claim 4.5 (c10) If \( \tilde{G} = Z_2 \), then**
\[ h^0(X, \omega_X \otimes P) = 1 \text{ for all } P \in \mathcal{S}_0. \]
Proof of Claim 4.5. It is clear that $h^0(\tilde{X}, \omega_{\tilde{X}} \otimes P) = h^0(A', \omega_{A'} \otimes M' \otimes P)$ for all $P \in S_\eta$, and $h^0(\tilde{X}, \omega_{\tilde{X}} \otimes P) = 1$ for general $P \in S_\eta$.

If $h^0(\tilde{X}, \omega_{\tilde{X}} \otimes Q_0) \geq 2$ for some $Q_0 \in S_\eta$, then $h^0(\tilde{X}, \omega_{\tilde{X}} \otimes Q_0) = 2$ as otherwise $h^0(\omega_{\tilde{X}}^{\otimes 2} \otimes Q_0^{\otimes 2}) \geq 3 + 3 - 1$ which is impossible.

Consider the linear series $|K_{A'} + M' + Q_0|$. Let $\mu : \tilde{A} \to A'$ be a log resolution of this linear series. We have

$$\mu^*|K_{A'} + M' + Q_0| = |D| + F,$$

where $|D|$ is base point free and $F$ has simple normal crossings support. There is an induced map $\phi|D| : \tilde{A} \to \mathbb{P}^1$ such that $|D| = \phi^*|O_{\mathbb{P}^1}(1)|$. We have an inclusion

$$\varphi_1 : \phi^*|O_{\mathbb{P}^1}(2)| + G \hookrightarrow \mu^*|2K_{A'} + 2M' + 2Q_0|.$$

For all $\eta \in \text{Pic}^0(Y)$, there is a morphism

$$\varphi_2 : \mu^*|K_{A'} + M' + Q_0 + \eta| + \mu^*|K_{A'} + M' + Q_0 - \eta| \to \mu^*|2K_{A'} + 2M' + 2Q_0|.$$

Notice that $h^0(A', \omega_{A'}^2 \otimes M' \otimes Q_0^{\otimes 2}) \leq h^0(X, \omega_X^2 \otimes Q_0^{\otimes 2}) \leq 4$. Since $h^0(\mathbb{P}^1, O_{\mathbb{P}^1}(2)) = 3$, $\varphi_1$ has a 2-dimensional image. Since $\eta$ varies in a 2-dimensional family, $\varphi_2$ also has 2-dimensional image. In particular, there is a positive dimensional family $N \subset \text{Pic}^0(Y)$ such that for general $\eta \in N$, one has

$$D_{\pm\eta} + F_{\pm\eta} \in \mu^*|K_{A'} + M' + Q_0 \pm \eta|$$

where $G = F_\eta + F_{-\eta}$ and $D_\eta + D_{-\eta} \in \phi^*|O_{\mathbb{P}^1}(2)|$. Since $G$ is a fixed divisor, it decomposes in at most finitely many ways as the sum of two effective divisors and so we may assume that $F_\eta, F_{-\eta}$ do not depend on $\eta \in N$.

Take any $\eta \neq \eta' \in N$ with $F_\eta = F_{\eta'}$. One has that $D_\eta = \phi^*|H|$ is numerically equivalent to $D_{\eta'} = \phi^*|H'|$. It follows that $H$ and $H'$ are numerically equivalent on $\mathbb{P}^1$ hence linearly equivalent. Thus $D_\eta$ and $D_{\eta'}$ are linearly equivalent which is a contradiction.

\[ \square \]

Claim 4.6 (c11). If $\tilde{G} = \mathbb{Z}_2$, then a : $X \to A$ has generic degree 2 and is branched over a divisor $B \in [2f^*\Theta]$ where $O_Y(\Theta)$ is an ample line bundle of degree 1. Furthermore, $a_* (O_X) \cong O_A \oplus q^*O_Y(\Theta) \otimes P$ where $P \notin \text{Pic}^0(Y)$ and $P^{\otimes 2} = O_A$.

See Example 2.

Proof of Claim 4.6. For all $\eta \in \text{Pic}^0(Y)$ and $P \in S_\eta$, one has that

$$h^0(\omega_X \otimes P \otimes \eta) = h^0(\omega_X \otimes M' \otimes P \otimes \eta) = 1.$$ 

The sheaf $q_*\nu_*(\omega_{A'} \otimes M' \otimes P)$ is torsion free of generic rank 1 and

$$h^0(q_*\nu_*(\omega_{A'} \otimes M' \otimes P) \otimes \eta) = 1 \quad \text{for all } \eta \in \text{Pic}^0(Y).$$

Following the proof of Proposition 4.2 of [HP], one sees that higher cohomologies vanish. By [Hac1], $q_*\nu_*(\omega_{A'} \otimes M' \otimes P)$ is a principal polarization $O_Y(\Theta)$. From the isomorphism $\nu_*(\omega_{A'} \otimes M' \otimes P) \cong L \otimes I(\tilde{B}/2)$, one sees that $L = O_Y(\Theta)$ and $I(\tilde{B}/2) = O_Y$. Therefore, $\nu_*(\omega_{A'} \otimes M' \otimes P) \cong q^*O_Y(\Theta)$. It follows that

$$a_* (\omega_X) \cong O_A \oplus q^*O_Y(\Theta) \otimes P.$$ 

\[ \square \]

From now on we therefore assume that $\tilde{G} \neq \mathbb{Z}_2$.

Claim 4.7 (cl7). $V^0(K_X)$ has at most one 2-dimensional component.
Proof of Claim 4.7. Let \( S_\eta, S_\zeta \) be 2-dimensional components of \( V^0(\omega_X) \) with \( \eta \neq \zeta \). Since \( \kappa(X) = 2 \), one has \( \delta_{\eta,\zeta} = 2 \). Thus by (2), \( P_{2,\eta+\zeta} \geq 3 \). By Lemma 3.4, this is impossible. \( \square \)

Claim 4.8 (cl8). Let \( T_1, T_2 \) be two parallel 1-dimensional components of \( V^0(\omega_X) \), then \( T_1 + \text{Pic}^0(Y) = T_2 + \text{Pic}^0(Y) \).

Proof of Claim 4.8. Let \( P_i \in T_i \), \( \pi : X \rightarrow E := T_1^\vee = T_2^\vee \) the induced morphism and \( L_i \) ample line bundles on \( E_i \) with inclusions \( \phi_i : \pi^*L_i \rightarrow \omega_X \otimes P_i \). By Lemma 2.12, one sees that \( h^0(\omega_X ^\otimes \otimes P_1 \otimes P_2) \geq 2 \). If it were equal, then the inclusion \( L_1 \otimes L_2 \rightarrow \pi_*((\omega_X ^\otimes \otimes P_1 \otimes P_2)) \) would be an I.T. 0 isomorphisms and this would imply that \( P_2(X/E) = 1 \) and hence that \( \kappa(X) \leq 1 \). So \( h^0(\omega_X ^\otimes \otimes P_1 \otimes P_2) \geq 3 \). By Lemma 3.4, this is impossible. \( \square \)

Claim 4.9 (cl12). If \( G \neq \mathbb{Z}_2 \), let \( S_\eta \) be a 2-dimensional component of \( V^0(\omega_X) \), then \( h^0(\omega_X \otimes \mathcal{P}) = 1 \) for all \( \mathcal{P} \in S_\eta \). In particular \( f_*(\omega_X \otimes \mathcal{P}) \) is a principal polarization.

Proof of Claim 4.9. Let \( f : X \rightarrow (S_\eta)^\vee \) be the induced morphism. Then \( f \) is birational to the Iitaka fibration of \( X \). By Claim 4.7, \( V^0(\omega_X) \) has at most one 2-dimensional component, and so there must exist a 1-dimensional component \( S_\zeta \) of \( V^0(\omega_X) \). Let \( \pi : X \rightarrow E := T_\zeta^\vee \) be the induced morphism. There is an ample line bundle \( L \) on \( E \) and an inclusion \( \pi^*L \rightarrow \omega_X \otimes Q_\zeta \) for some general \( Q_\zeta \in S_\zeta \).

Assume that \( \mathcal{P} \in S_\eta \) and \( h^0(\omega_X \otimes \mathcal{P}) \geq 2 \). If \( \text{rank}(\pi_*(\omega_X \otimes \mathcal{P})) = 1 \), then \( \pi_*(\omega_X \otimes \mathcal{P}) \) is an ample line bundle of degree at least 2 and hence \( h^0(\pi_*(\omega_X \otimes \mathcal{P}) \otimes \eta) \geq 2 \) for all \( \eta \in \text{Pic}^0(E) \). It follows that

\[
h^0(\omega_X ^\otimes \otimes \mathcal{P} \otimes Q_\zeta) \geq h^0(\omega_X \otimes \pi^*L) = h^0(\pi_*(\omega_X \otimes \mathcal{P}) \otimes L) \geq \text{deg}(\pi_*(\omega_X \otimes \mathcal{P})) + \text{deg}(\pi_*(\omega_X \otimes \mathcal{P})) \geq 1 + 2 = 3.
\]

By Lemma 3.4, this is impossible. Therefore, we may assume that \( \text{rank}(\pi_*(\omega_X \otimes \mathcal{P})) \geq 2 \). Proceeding as above \( \pi_*(\omega_X \otimes \mathcal{P}) \) is a sheaf of degree at least 0. Since \( h^0(\pi_*(\omega_X \otimes \mathcal{P}) \otimes \eta) > 0 \) for all \( \eta \in \text{Pic}^0(E) \), By Riemann-Roch one sees that also \( h^1(\pi_*(\omega_X \otimes \mathcal{P}) \otimes \eta) > 0 \) for all \( \eta \in \text{Pic}^0(E) \). By Theorem 2.5, this is impossible.

Finally, the sheaf \( f_*(\omega_X \otimes \mathcal{P}) \) is torsion free of generic rank 1 on \( Y \) and hence, by [Hac1], it is a principal polarization. \( \square \)

Claim 4.10 (cl13). Assume that \( G \neq \mathbb{Z}_2 \). Then, for any \( \mathcal{P} \in V^0(\omega_X) - \text{Pic}^0(Y) \) one has that \( f_*(\omega_X \otimes \mathcal{P}) \) is either:

i) a principal polarization on \( Y \);
ii) the pull-back of a line bundle of degree 1 on an elliptic curve or
iii) of the form \( L \otimes L' \otimes \mathcal{I}_p \) where \( L, L' \) are ample line bundles of degree 1 on \( E, E' \), \( Y = E \times E' \) and \( p \) is a point of \( Y \).

In particular, there are no 2 distinct parallel components of \( V^0(\omega_X) \).

Proof of Claim 4.10. By Claim 4.9, we only need to consider the case in which all the components of \( (P + \text{Pic}^0(Y)) \cap V^0(\omega_X) \) are 1-dimensional. By Claim 4.3, we may also assume that these components are parallel.

For any 1 dimensional component \( T_i \) of \( (P + \text{Pic}^0(Y)) \cap V^0(\omega_X) \), \( P_i \in T_i \) and corresponding projection \( \pi_i : X \rightarrow E_i := T_i^\vee \), one has \( \text{rank}(\pi_i_*(\omega_X \otimes P_i)) = 1 \) and hence \( \pi_1_*(\omega_X \otimes P_i) \) is an ample line bundle of degree at least 1 on \( E_i \). If this were not the case, then By Lemma 2.13,

\[
\text{rank}(\pi_i_*(\omega_X \otimes P_i)) = h^0(\omega_F) \geq 2
\]

and so

\[
\text{rank}(\pi_i_*(\omega_X ^\otimes \otimes P_i)) = h^0(\omega_F ^\otimes \otimes P_i) \geq 3.
\]
From the inclusion (cf. Corollary 2.11)

\[ \pi_i^*L_i \hookrightarrow \omega_X \otimes P_i \hookrightarrow \omega_X^{\oplus 2} \otimes P_i, \]

one sees that \( h^0(\omega_X^{\oplus 2} \otimes P_i) \geq 2 \) (cf. Lemma 3.1). By Lemma 2.4, \( \deg(\pi_i, (\omega_X^{\oplus 2} \otimes P_i)) \geq 2 \). By Riemann-Roch, one has

\[ h^0(L \otimes \pi_i, (\omega_X^{\oplus 2} \otimes P_i)) \geq \deg(\pi_i, (\omega_X^{\oplus 2} \otimes P_i)) + \text{rank}(\pi_i, (\omega_X^{\oplus 2} \otimes P_i)) \geq 5. \]

This is a contradiction and so \( \text{rank}(\pi_i, (\omega_X^{\oplus 2} \otimes P_i)) = 1 \).

Since we assumed that all components of \( V^0(\omega_X) \cap (P + \text{Pic}^0(Y)) \) are parallel, then one has \( \pi_i = \pi, E = E_i \) are independent of \( i \). Let \( q : Y \hookrightarrow E \). Since there are injections

\[ \text{Pic}^0(Y) + P_1 = T_1 \hookrightarrow P_1 + \text{Pic}^0(Y) \hookrightarrow \text{Pic}^0(X), \]

we may assume that \( q \) has connected fibers. The sheaf \( f_*(\omega_X \otimes P_i) \) is torsion free of rank 1, and hence we may write \( f_*(\omega_X \otimes P_i) \cong M \otimes \mathcal{I} \) where \( M \) is a line bundle and \( \mathcal{I} \) is supported in codimension at least 2 (i.e. on points). Since \( \text{rank}(\pi_*, (\omega_X \otimes P_i)) = 1 \), one has that \( h^0(M|_{F_Y/E}) = 1 \).

For general \( \eta \in \text{Pic}^0(Y) \), one has that \( V^0(\omega_X) \cap P_1 + \eta + \text{Pic}^0(E) = \emptyset \) and so the semipositive torsion free sheaf \( \pi_* (\omega_X \otimes P_1 \otimes \eta) \) must be the 0-sheaf. In particular \( h^0(M \otimes \eta|_{F_Y/E}) = 0 \). It follows that \( \deg(M|_{F_Y/E}) = 0 \) and hence \( M|_{F_Y/E} = \mathcal{O}_{F_Y/E} \).

One easily sees that \( h^0(M \otimes \eta) = 0 \) for all \( \eta \in \text{Pic}^0(Y) - \text{Pic}^0(E) \) and hence

\[ V^0(\omega_X) = P_1 + \text{Pic}^0(E) = T_1. \]

By Proposition 2.3, one has that \( q^*L_1 \) and \( f_*(\omega_X \otimes P_i) \) are isomorphic if and only if the inclusion \( q^*L_1 \hookrightarrow f_*(\omega_X \otimes P_i) \) induces isomorphisms

\[ H^i(Y, q^*L_1 \otimes \eta) \hookrightarrow H^i(Y, f_*(\omega_X \otimes P_i) \otimes \eta) \]

for \( i = 0, 1, 2 \) and all \( \eta \in \text{Pic}^0(Y) \). If \( \eta \in \text{Pic}^0(Y) - \text{Pic}^0(E) \) or if \( i = 2 \) and \( \eta \in \text{Pic}^0(E), \) then both groups vanish and so the isomorphism follows. If \( \eta \in \text{Pic}^0(E) \) and \( i = 0 \), then the isomorphism follows as

\[ H^0(Y, q^*L_1 \otimes \eta) = H^0(E, L_1 \otimes \eta) = H^0(E, \pi_* (\omega_X \otimes P_1) \otimes \eta) = H^0(Y, f_*(\omega_X \otimes P_1) \otimes \eta). \]

If \( i = 1 \) and \( \eta \in \text{Pic}^0(E), \) we remark that by Theorem 2.5 c) and e), for any \( v \in H^1(Y, \mathcal{O}_Y) \) which is not tangent to \( \text{Pic}^0(E), \) one has an isomorphism

\[ H^0(Y, f_*(\omega_X \otimes P_1) \otimes \eta) \xrightarrow{\cup v} H^1(Y, f_*(\omega_X \otimes P_1) \otimes \eta). \]

Since

\[ H^0(Y, q^*L_1 \otimes \eta) \xrightarrow{\cup v} H^1(Y, q^*L_1 \otimes \eta) \]

is also an isomorphism, the statement follows.

\[ \square \]

Claim 4.11 (cl14). If \( \bar{G} \neq \mathbb{Z}_2 \), then \( \bar{G} \cong (\mathbb{Z}_2)^2 \) and \( V^0(X, \omega_X) \) contains a 2-dimensional component.

Proof of Claim 4.11. We have seen that \( V^0(\omega_X) \) has at most one 2-dimensional component and there are no parallel 1-dimensional components. Since \( \bar{G} \neq \mathbb{Z}_2 \), then there are at least two 1-dimensional components of \( V^0(\omega_X) \). We will show that given two one dimensional components contained in \( Q_1 + \text{Pic}^0(Y) \neq Q_2 + \text{Pic}^0(Y) \), then

\[ (Q_1 + Q_2 + \text{Pic}^0(Y)) \cap V^0(\omega_X) \]

does not contain a 1-dimensional component. By Proposition 2.7, it follows that \( Q_1 + Q_2 + \text{Pic}^0(Y) \) is a 2-dimensional component of \( V^0(\omega_X) \). If \( |\bar{G}| > 4 \), this implies that there are at least two 2-dimensional components, which is impossible, so \( \bar{G} = (\mathbb{Z}_2)^2 \) and the claim follows. \[ \square \]
Claim 4.12 (c16). If $G \not\cong \mathbb{Z}_2$, then $V_0(X, \omega_X) = \{O_X\} \cup S_\eta \cup S_\zeta \cup S_\xi$ with $\delta_\eta = 2$, $\delta_\zeta = \delta_\xi = 1$.

Proof of Claim 4.12. Suppose that there are three 1-dimensional components of $V^0(\omega_X)$, say $S_1, S_2, S_3$, contained in $Q_1 + \text{Pic}^0(Y), Q_2 + \text{Pic}^0(Y), Q_3 + \text{Pic}^0(Y)$ respectively with $Q_1 + Q_2 + Q_3 \in \text{Pic}^0(Y)$. By Claim 4.10, these components are not parallel to each other. We may assume that $\pi_i : X \to E_i := S_i \cdot Y$ factors through $f : X \to Y$ and that $Y$ is an abelian surface. Let $q_i : Y \to E_i$ be the induced morphisms.

Let $Q_1, Q_2, Q_3$ be general elements in $S_1, S_2, S_3$ and

$$G := f_*(\omega_X^{\otimes 2} \otimes Q_2 \otimes Q_3), \quad F := f_*(\omega_X^{\otimes 3} \otimes Q_1 \otimes Q_2 \otimes Q_3).$$

From the inclusions $\pi_i^* L_i \to \omega_X \otimes Q_i$, one sees that we have inclusions

$$\varphi : q_2^* L_2 \otimes q_3^* L_3 \to G, \quad \psi : q_1^* L_1 \otimes q_2^* L_2 \otimes q_3^* L_3 \to F$$

where $L_i$ are ample line bundles on $E_i$ respectively. Since $F$ is torsion free of generic rank one, we may write

$$F = q_1^* L_1 \otimes q_2^* L_2 \otimes q_3^* L_3 \otimes N \otimes I$$

where $N$ is a semi-positive line bundle on $Y$ and $I$ is an ideal sheaf cosupported at points. If $N$ is not numerically trivial, then $N$ is not vertical with respect to one of the projections $q_i$, say $q_1$. Then

$$\text{rank}(q_1, *)(F) = F_{Y/E_1} \cdot (q_1^* L_1 + q_2^* L_2 + q_3^* L_3 + N) \geq 3.$$  

On the other hand, from the inclusion $\varphi$, one sees that $\text{rank}(q_1, *)(G) \geq 2$. Consider the inclusion of I.T. 0 sheaves $L_1 \to q_1, *(G \otimes \eta)$ with $\eta = Q_1 \otimes Q_2 \otimes Q_3 \in \text{Pic}^0(Y)$. Since it is not an isomorphism, one sees that

$$h^0(G) = h^0(G \otimes \eta) > h^0(L_1) \geq 1.$$  

From the inclusion

$$\rho : L_1 \otimes q_1, *(G) \to q_1, *(F) = \pi_1, *(\omega_X^{\otimes 3} \otimes Q_1 \otimes Q_2 \otimes Q_3)$$

one sees that by Riemann-Roch

$$h^0(G) + \text{rank}(q_1, *)(G) \leq h^0(\omega_X^{\otimes 3} \otimes Q_1 \otimes Q_2 \otimes Q_3) = P_3(X)$$

and therefore

$$h^0(G) = 2, \quad \text{rank}(q_1, *)(G) = 2.$$  

In particular, $\rho$ is an I.T. 0 isomorphism. So, $\text{rank}(q_1, *)(F) = \text{rank}(q_1, *)(G) = 2$ which is a contradiction. Therefore, we have that $N \in \text{Pic}^0(Y)$ and $q_2^* L_2, F_{Y/E_1} = q_3^* L_3, F_{Y/E_1} = 1$. Recall that $\deg(L_i) = 1$ and so $q_1^* L_1 = F_{Y/E_1}$. Since $(q_1^* L_1 \otimes q_2^* L_2 \otimes q_3^* L_3)^2 \geq 8$, we have that $q_2^* L_2 : q_3^* L_3 \geq 2$. Since

$$h^0(q_2^* L_2 \otimes q_3^* L_3) \leq h^0(G) = 2,$$

one sees that $q_2^* L_2, q_3^* L_3 = 2$ and hence $I = O_Y$.

Now let $G' := f_*(\omega_X^{\otimes 2} \otimes Q_1 \otimes Q_3)$. Proceeding as above, one sees that

$$\text{rank}(q_2, *)(G') \geq F_{Y/E_2} \cdot (q_1^* L_1 + q_2^* L_3) = 3, \quad h^0(q_2, *, G') > h^0(L_2) = 1.$$  

By Riemann Roch, one has that

$$P_3(X) = h^0(\omega_X^{\otimes 3} \otimes Q_1 \otimes Q_2 \otimes Q_3) \geq h^0(L_1 \otimes q_2, *, G') \geq 5$$

and hence $\delta_{-\eta-\zeta} = 1$. In particular, there is a 1-dimensional component of $V^0(\omega_X) \cap \text{Pic}^0(Y) - \eta - \zeta$.

Let $\pi : X \to E := R^Y$ be the induced morphism, then there is an ample line bundle $L$ on $E$ and an inclusion $L \to \pi_*(\omega_X \otimes Q_{-\eta} \otimes Q_{-\zeta})$.

By Corollary 3.2, one has $P_3 \geq 2 + P_{2, \eta+\zeta} \geq 5$ which is the required contradiction.
which is the required contradiction.

\begin{proof}[Proof of Claim 4.13] Let $G \cong \mathbb{Z}_2^2$, then $Y = E_1 \times E_2$ and there are line bundles $L_i$ of degree 1 on $E_i$, projections $p_i : A \rightarrow E_i$ and 2-torsion elements $Q_1, Q_2 \in \text{Pic}^0(X)$ that generate $G$, such that

\[ a_*(\mathcal{O}_X) \cong \mathcal{O}_A \oplus M_1' \oplus M_2' \oplus M_1'' \oplus M_2'' \]

with

\[ M_1 = p_1^* L_1 \otimes Q_1', \quad M_2 = p_2^* L_2 \otimes Q_2' \quad \text{and} \quad M_3 = M_1 \otimes M_2. \]

In particular $X$ is birational to the fiber product of two degree 2 coverings $X_i \rightarrow A$ with $P_3(X_i) = 2$.

By Claim 4.11 and Claim 4.12, the degree of $a : X \rightarrow A$ is $|\overline{G}| = 4$ and there are two non parallel 1-dimensional components of $V^0(\omega_X)$ say $S_1, S_2$ such that $S_1 + \text{Pic}^0(Y) \neq S_2 + \text{Pic}^0(Y)$. Let $E_1 := S_1^\vee$ and $q_i : Y \rightarrow E_i, \pi_i : X \rightarrow E_i$ be the induced morphisms. Then there are inclusions $\pi_i^* L_i \rightarrow \omega_X \otimes Q_i$ where $Q_i \in S_i$. Moreover, by Claim 4.12, $Q_1 + Q_2 + \text{Pic}^0(Y) \subset V^0(\omega_X)$. By Claim 4.9, one has that

\[ L := f_*(\omega_X \otimes Q_1 \otimes Q_2) \]

is an ample line bundle of degree 1. Moreover,

\[ V^0(\omega_X) = \{ \mathcal{O}_X \} \cup S_1 \cup S_2 \cup (Q_1 + Q_2 + \text{Pic}^0(Y)). \]

From the inclusion

\[ q_1^* L_1 \otimes q_2^* L_2 \otimes L \rightarrow f_*(\omega_X^{\otimes 3} \otimes Q_1^{\otimes 2} \otimes Q_2^{\otimes 2}) \]

and the equality $4 = P_3(X) = h^0(\omega_X^{\otimes 3} \otimes Q_1^{\otimes 2} \otimes Q_2^{\otimes 2})$, one sees that $L.L_i = L_1.L_2 = 1$. Therefore,

\[ L = q_1^* (L_1 \otimes P_1) \otimes q_2^* (L_2 \otimes P_2), \quad P_i \in \text{Pic}^0(E_i), \]

\[ (Y, q_1^* L_1 \otimes q_2^* L_2) \cong (E_1, L_1) \times (E_2, L_2). \]

We have inclusions

\[ L \rightarrow f_*(\omega_X \otimes Q_1 \otimes Q_2) \rightarrow f_*(\omega_X^{\otimes 2} \otimes Q_1 \otimes Q_2), \]

\[ q_1^* L_1 \otimes q_2^* L_2 \rightarrow f_*(\omega_X^{\otimes 2} \otimes Q_1 \otimes Q_2). \]

Let $G := \omega_X^{\otimes 2} \otimes Q_2 \otimes Q_2$. If $h^0(G) = 1$, then $L = q_1^* L_1 \otimes q_2^* L_2$ as required. If $h^0(G) \geq 2$, then one sees that

\[ h^0(\pi_1, G) \otimes L_1 \otimes P_1) \geq \text{rank}(G) + \text{deg}(G) \geq 1 + 2. \]

Since

\[ \text{rank}(\pi_2, (G \otimes \pi_1^*(L_1 \otimes P_1))) \geq \text{rank}(q_2, (q_1^*(L_1^{\otimes 2} \otimes P_1) \otimes q_2^*(L_2))) = 2, \]

one sees that

\[ P_3(X) \geq h^0(\omega_X^{\otimes 2} \otimes Q_1 \otimes Q_2 \otimes L) = h^0(\pi_2, (G \otimes \pi_1^*(L_1 \otimes P_1)) \otimes L_2 \otimes P_2) \geq 2 + 3 \]

and this is impossible. Let $M_i := p_i^* L_i \otimes Q_i'$. By Claim 4.10, one has

\[ a_*(\omega_X) \cong \mathcal{O}_A \oplus M_1' \oplus M_2' \oplus M_1'' \oplus M_2''. \]

and hence by Grothendieck duality,

\[ a_*(\mathcal{O}_X) \cong \mathcal{O}_A \oplus M_1' \oplus M_2' \oplus M_1'' \oplus M_2''. \]

Let $X \rightarrow A$ be the Stein factorization. Following [HM] §7, one sees that the only possible nonzero structure constants defining the 4-1 cover $Z \rightarrow A$ are $c_{1,4} \in H^3(M_1 \otimes M_2 \otimes M_1')$, $c_{1,6} \in H^0(M_1 \otimes M_2' \otimes M_3)$ and $c_{4,6} \in H^0(M_1' \otimes M_2 \otimes M_3)$. So, $Z \rightarrow A$ is a bi-double cover. It is determined by two degree 2 covers $a_i : X_i \rightarrow A$ defined by $a_{i,*}(\mathcal{O}_X) = \mathcal{O}_A \oplus p_i^* L_i \otimes Q_i'$ and sections $-c_{1,4} c_{1,6} \in H^0(M_1^{\otimes 2})$ and $c_{1,4} c_{4,6} \in H^0(M_2^{\otimes 2})$. It is easy to see that $X_1, X_2, Z$ are smooth. \qed
This completes the proof. □

5. Varieties with $P_3(X) = 4$, $q(X) = \dim(X)$ and $\kappa(X) = 1$

**Theorem 5.1 (main).** Let $X$ be a smooth projective variety with $P_3(X) = 4$, $q(X) = \dim(X)$ and $\kappa(X) = 1$ then $X$ is birational to $(C \times \bar{K})/G$ where $G$ is an abelian group acting faithfully by translations on an abelian variety $\bar{K}$ and faithfully on a curve $C$. The Iitaka fibration of $X$ is birational to $f : (C \times \bar{K})/G \to C/G = E$ where $E$ is an elliptic curve and $\dim H^0(C, \omega_C^{\otimes 3})^G = 4$.

**Proof.** Let $f : X \to Y$ be the Iitaka fibration. Since $\kappa(X) = 1$, and $\alpha : X \to A$ is generically finite, one has that $Y$ is a curve of genus $g \geq 1$. If $g = 1$, then $Y$ is an elliptic curve and $Y \to A(Y)$ is an étale map. By the universal properties of the Albanese morphism of $X$, one sees that $Y \to A(Y)$ is of degree 1 (i.e. an isomorphism). By Proposition 2.1 one sees that if $g \geq 2$, then $q(X) \geq \dim(X) + 1$ which is impossible.

From now on we will denote the elliptic curve $A(Y)$ simply by $E$ and $f : X \to E$ will be the corresponding algebraic fiber space. Let $X \to \bar{X} \to A$ be the Stein factorization of the Albanese map. Since $X \to A$ is isotrivial, there is a generically finite cover $C \to E$ such that $\bar{X} \times_E C$ is birational to $C \times \bar{K}$. We may assume that $C \to E$ is a Galois cover with group $G$. $G$ acts by translations on $\bar{K}$ and we may assume that the action of $G$ is faithful on $C$ and $\bar{K}$. Since $G$ acts freely on $C \times \bar{K}$, one has that

$$H^0(X, \omega_X^{\otimes 3}) = H^0(C \times \bar{K}, \omega_{C \times \bar{K}}^{\otimes 3})^G = [H^0(\bar{K}, \omega_{\bar{K}}^{\otimes 3}) \otimes H^0(C, \omega_C^{\otimes 3})]^G.$$ 

Since $G$ acts on $\bar{K}$ by translations, $G$ acts on $H^0(\bar{K}, \omega_{\bar{K}}^{\otimes 3})$ trivially. It follows that

$$4 = P_3(X) = \dim H^0(C, \omega_C^{\otimes 3})^G.$$ 

Similarly, one sees that $q(X) = q(C/G) + q(\bar{K}/G)$ and so $q(C/G) = 1$. □

We now consider the induced morphism $\pi : C \to C/G =: E$. By the argument of [Be], Example VI.12, one has

$$4 = \dim H^0(C, \omega_C^{\otimes 3})^G = h^0(E, \mathcal{O}(\sum_{P \in E} [3(1 - \frac{1}{e_P})])).$$

Where $P$ is a branch points of $\pi$, and $e_P$ is the ramification index of a ramification point lying over $P$. Note that $|G| = e_P s_P$, where $s_P$ is the number of ramification points lying over $P$.

It is easy to see that since

$$\left\lceil 3(1 - \frac{1}{e_P}) \right\rceil = 1 \text{ (resp. } 2\right\rceil$$ if $e_P = 2$ (resp. $e_P \geq 3$),

we have the following cases:

**Case 1.** 4 branch points $P_1, \ldots, P_4$ with $e_{P_1} = 2$.

**Case 2.** 3 branch points $P_1, P_2, P_3$ with $e_{P_1} \geq 3, e_{P_2} = e_{P_3} = 2$.

**Case 3.** 2 branch points $P_1, P_2$ with $e_{P_1} \geq 3$.

We will follow the notation of [Pa]. Let $\pi : C \to E$ be an abelian cover with abelian Galois group $G$. There is a splitting

$$\pi_* \mathcal{O}_C = \bigoplus_{\chi \in G} L^\chi.$$ 

In particular, if $d_\chi := \deg(L^\chi)$, then

$$g = 1 + \sum_{\chi \in G^s, \chi \neq 1} d_\chi.$$
For every branch point \( P_i \) with \( i = 1, \ldots, s \), the inertia group \( H_i \), which is defined as the stabilizer subgroup at any point lying over \( P_i \), is a cyclic subgroup of order \( e_i := e_{P_i} \). We also associate a generator \( \psi_i \) of each \( H_i^* \) which corresponds to the character of \( P_i \). For every \( \chi \in G^* \), \( \chi|H = \psi_i^{n(\chi)} \) with \( 0 \leq n(\chi) \leq |H| - 1 \). And define

\[
\epsilon_{\chi, \chi'}^{H, \psi_i} := \left\lfloor \frac{n(\chi) + n(\chi')}{|H|} \right\rfloor.
\]

Following [Pa], one sees that there is an abelian cover \( C \rightarrow E \) with group \( G \) with building data \( L_\chi \) if and only if the line bundles \( L_\chi \) satisfy the following set of linear equivalences:

\[ [\text{bundle}] L_\chi + L_{\chi'} = L_{\chi \chi'} + \sum_{i=1, \ldots, s} \epsilon_{\chi, \chi'}^{H, \psi_i} P_i. \]

If \( \chi|H_i = \psi_i^{n(\chi)} \), then

\[ [\text{degree}] d_\chi + d_{\chi'} = d_{\chi \chi'} + \sum_{i=1, \ldots, s} \left| \frac{n_i(\chi) + n_i(\chi')}{e_i} \right|. \]

Let \( H \) be the subgroup of \( G \) generated by the inertia subgroups \( H_i \) and let \( Q = G/H \). One sees that there is an exact sequence of groups

\[ 1 \rightarrow Q^* \rightarrow G^* \rightarrow H^* \rightarrow 1. \]

The generators \( \psi_i \) of \( H_i^* \) define isomorphisms \( H_i^* \cong \mathbb{Z}_{e_i} \), where \( e_i := |H_i| \). Therefore, we have an induced injective homomorphism

\[ \varphi : H^* \hookrightarrow \prod_{i=1, \ldots, s} \mathbb{Z}_{e_i} \]

such that the induced maps \( \varphi_i : H^* \rightarrow \mathbb{Z}_{e_i} \) are surjective. By abuse of notation, we will also denote by \( \varphi \) the induced homomorphism \( \varphi : G^* \rightarrow \prod_{i=1, \ldots, s} \mathbb{Z}_{e_i} \). We will write

\[ \varphi(\chi) = (n_1(\chi), \ldots, n_s(\chi)) \quad \forall \chi \in G^*. \]

Let \( \mu(\chi) \) be the order of \( \chi \). By [Pa] Proposition 2.1,

\[ d_\chi = \sum_{i=1, \ldots, s} \frac{n_i(\chi)}{e_i}. \]

We will now analyze all possible inertia groups \( H \).

**Case 1:** \( s = 4 \) and \( e := e_i = 2 \). Then \( H^* \subset \mathbb{Z}_2^4 \).

Note that \( H^* \neq \mathbb{Z}_2^4 \) since \((1, 0, 0, 0) \notin H^* \). Thus \( H^* \cong (\mathbb{Z}_2)^s \) with \( 1 \leq s \leq 3 \).

By Example 1, all of these possibilities occur.

**Case 2:** \( s = 3 \) and \( e_1 \geq 3 \), \( e_2 = e_3 = 2 \).

There must be a character \( \chi \) with \( \varphi(\chi) = (1, n_2, n_3) \), and so

\[ d_\chi = \frac{1}{e_1} + \frac{n_2}{2} + \frac{n_3}{2}, \]

which is not an integer. Therefore this case is impossible.

**Case 3:** \( s = 2 \) and \( e_1, e_2 \geq 3 \).

Assume that \( e_1 > e_2 \). Since \( G^* \rightarrow \mathbb{Z}_{e_1} \) is surjective, there is \( \chi \in H^* \) with \( \varphi(\chi) = (1, n_2) \). Then

\[ d_\chi = \frac{1}{e_1} + \frac{n_2}{e_2} < 1 \]

which is impossible. So we may assume that \( e = e_1 = e_2 \geq 3 \) and \( H^* \subset \mathbb{Z}_2^e \). Let \( \varphi(\chi) = (n_1, n_2) \). One has \( d_\chi = \frac{n_1 + n_2}{e} \). Thus \( n_2 = e - n_1 \) for any \( \chi \neq 1 \). Therefore, \( H^* = \{(i, e-i)|0 \leq i \leq e-1\} \cong \mathbb{Z}_e \). By Example 1, all of these possibilities occur.

From the above discussion, it follows that:
Proposition 5.2 (cover). Let $\phi : C \rightarrow E$ be a $G$-cover with $E$ an elliptic curve and $\dim H^0(\omega_C^3)^G = 4$. Then either $\phi$ is ramified over 4-points and the inertia group $H$ is isomorphic to $(\mathbb{Z}_2)^s$ with $s \in \{1, 2, 3\}$ or $\phi$ is ramified over 2-points and the inertia group $H$ is isomorphic to $\mathbb{Z}_m$ with $m \geq 3$.

REFERENCES


ALFRED JUNGKAI CHEN AND CHRISTOPHER D. HACON

Jungkai Alfred Chen
Department of Mathematics
National Taiwan University
No.1, Sec. 4, Roosevelt Road
Taipei, 106, Taiwan
jkchen@math.ntu.edu.tw

Christopher D. Hacon
Department of Mathematics
University of Utah
155 South 1400 East, JWB 233
Salt Lake City, Utah 84112-0090, USA
hacon@math.ucr.edu