Advanced Algebra II

TRANSCENDENTAL EXTENSION

We now start our discussion on transcendental extension. The main purpose is to show that the concept of transcendental degree, which is the cardinality of transcendental basis, can be well-defined. Moreover, transcendental degree is a good candidate for defining dimension.

Definition 0.1. Let $F/K$ be an extension. $S \subset F$ is said to be algebraically dependent (over $K$) if there is an $n \geq 1$ and an $f \neq 0 \in K[x_1, ..., x_n]$ such that $f(s_1, ..., s_n) = 0$ for some $s_1, ..., s_n$. Roughly speaking, some element of $S$ satisfy a non-zero algebraic relation $f$ over $K$.

$S$ is said to be algebraically independent over $K$ if it’s not algebraically dependent over $K$.

Example 0.2. For any $u \in F$, $\{u\}$ is algebraically dependent over $K$ if and only if $u$ is algebraic over $K$.

Example 0.3. In the extension $K(x_1, ..., x_n)/K$, $S = \{x_1, ..., x_n\}$ is algebraically independent over $K$.

The following theorem says that finitely generated purely transcendental extension are just rational function fields.

Theorem 0.4. If $\{s_1, ..., s_n\} \subset F$ is algebraically independent over $K$. Then $K(s_1, ..., s_n) \cong K(x_1, ..., x_n)$.

Proof. We consider the homomorphism $\theta : K[x_1, ..., x_n] \rightarrow K[s_1, ..., s_n]$. $\theta$ is surjective by definition. It’s injective because $\{s_1, ..., s_n\} \subset F$ is algebraically independent. Then $\theta$ induces an isomorphism on quotient fields. \qed

One notices that the notion of being algebraic independent is an analogue of being linearly independent. Therefore, one can try to define the notion of ”basis” and ”dimension” in a similar way.

Definition 0.5. $S \subset F$ is said to be a transcendental basis of $F/K$ if $S$ is a maximal algebraically independent set. In other words, for all $u \in F - S$, $S \cup \{u\}$ is algebraically dependent.

We will then define the transcendental degree to be the cardinality of a transcendental basis (in a analogue of dimension). In order to show that this is well-defined. We need to work harder.

Proposition 0.6. Let $S \subset F$ be an algebraically independent set over $K$ and $u \in F - K(S)$. Then $S \cup \{u\}$ is algebraically independent if and only if $u$ is transcendental over $K(S)$.

Proof. The proof is straightforward. \qed
Corollary 0.7. \( S \) is a transcendental basis of \( F/K \) if and only if \( F/K(S) \) is algebraic.

Proof. Suppose that \( S \) is a transcendental basis of \( F/K \). If \( u \in F - K(S) \), then \( S \cup \{ u \} \) is not algebraically independent. Thus, \( u \) is algebraic over \( K(S) \) by the Proposition.

On the other hand, suppose that \( F/K(S) \) is algebraic. Then for all \( u \in F - S \), \( u \) is algebraic over \( K(S) \). By the Proposition, \( S \cup \{ u \} \) is algebraically dependent if \( u \in F - K(S) \). In fact, it’s easy to see directly that \( S \cup \{ u \} \) is algebraically dependent if \( u \in K(S) \). Thus \( S \) is a maximal algebraically independent set. \( \square \)

Corollary 0.8. Let \( S \subset F \) be an subset over such that \( F/K(S) \) is algebraic. Then \( S \) contains a transcendental basis.

Proof. By Zorn’s Lemma, there exists a maximal algebraically independent subset \( S' \subset S \). Then \( K(S) \) is algebraic over \( K(S') \) and hence \( F \) is algebraic over \( K(S') \). \( \square \)

Theorem 0.9. Let \( S, T \) be transcendental bases of \( F/K \). If \( S \) is finite, then \(|T| = |S|\).

Proof. Let \( S = \{s_1,\ldots,s_n\} \) and \( S' := \{s_2,\ldots,s_n\} \). We first claim that there is an element \( t \in T \), say \( t = t_1 \) such that \( \{t_1, s_2, \ldots, s_n\} \) is a transcendental basis.

To see this, if every element of \( T \) is algebraic over \( K(S') \), then \( F \) is algebraic over \( K(T) \) hence over \( K(S) \) which is a contradiction. Thus, there is an element \( t \in T \), say \( t = t_1 \) such that \( t_1 \) is transcendental over \( K(S') \). And hence \( T' := \{t_1, s_2, \ldots, s_n\} \) is algebraically independent.

By the maximality of \( S \), one sees that \( s_1 \) is algebraic over \( K(T') \). It follows that \( F \) is algebraic over \( K(t_1, s_1, \ldots, s_n) \) and hence algebraic over \( K(T') \). Therefore, \( T' \) is a transcendental basis.

By induction, one sees that there is a transcendental basis \( \{t_1, \ldots, t_n\} \subset T \). Thus \( T = \{t_1, \ldots, t_n\} \). \( \square \)

Theorem 0.10. Let \( S, T \) be transcendental bases of \( F/K \). If \( S \) is infinite, then \(|T| = |S|\).

Proof. By the previous theorem, we may assume that \( T \) is finite as well.

For \( s \in S \), we have \( s \in F \) hence algebraic over \( K(T) \). Let \( T_s \subset T \) be the subset of \( T \) of elements that appearing in the minimal polynomial of \( s \). It’s clear that \( T_s \neq \emptyset \) otherwise, \( s \) is algebraic \( K \) which is not the case. Also note that \( T_s \) is finite.

Let \( T' := \cup_{s \in S} T_s \). We claim that \( T' = T \). To this end, one sees that for \( u \in F \), \( u \) is algebraic over \( K(S) \) and hence algebraic over \( K(T') \). Thus \( F/K(T') \) is algebraic. \( T \) is a transcendental basis, hence \( T = T' \).

Lastly, one sees that
\[
|T| = |T'| = |\cup_{s \in S} T_s| \leq |S| \cdot \aleph_0 = |S|.
\]
Replace $S$ by $T$, one has $|S| \leq |T|$. We are done. \hfill \Box

With these two theorem, we can define the transcendental degree of an extension. And the definition is independent of choices of basis.

**Definition 0.11.** Let $F/K$ be an extension and $S$ be a transcendental basis. We define the transcendental degree of $F/K$, denoted $\text{tr.d.} F/K$, to be $|S|$.

**Theorem 0.12.** Let $F/E$ and $E/K$ be extensions. Then

$$\text{tr.d.} F/K = \text{tr.d.} F/E + \text{tr.d.} E/K.$$  

**Proof.** Let $T$ be a transcendental basis of $F/E$ and $S$ be a transcendental basis of $E/K$. We would like to show that $S \cup T$ is a transcendental basis of $F/K$. Note that $T \cap E = \emptyset$, hence $S \cap T = \emptyset$. Thus $|S \cup T| = |S| + |T|$, and we are done.

To see the claim, it’s easy to check that $E(T) = EK(S \cup T)$. Hence $E(T)/K(S \cup T)$ is algebraic if $E/K(S)$ is algebraic. Also, $F/E(T)$ is algebraic, therefore, $F/K(S \cup T)$ is algebraic.

It suffices to show that $S \cup T$ is algebraically independent. Suppose that there is $f(x_1, \ldots, x_n, y_1, \ldots, y_m)$ such that $f(s_1, \ldots, s_n, t_1, \ldots, t_m) = 0$. We can write

$$f(x_1, \ldots, x_n, y_1, \ldots, y_m) = \sum_I h_I(x_1, \ldots, x_n)y^I,$$

and we have $\sum_I h_I(s_1, \ldots, s_n)t^I$. Since $T$ is algebraically independent over $E \ni h_I(s_1, \ldots, s_n)$. It follows that $h_I(s_1, \ldots, s_n) = 0$ for all $I$. Since $S$ is algebraically independent over $K$, if follows that $h_I(x_1, \ldots, x_n) = 0 \in K[x_1, \ldots, x_n]$ for all $I$. Therefore $f(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0$. Hence $S \cup T$ is algebraically independent. \hfill \Box