Advanced Algebra II

Localization

Before we going to the study of dimension theory, we need to recall some basic notion of localization.

**Definition 0.1.** A subset $S \subset R$ is said to be a multiplicative set if
   
   (1) $1 \in S$,
   
   (2) if $a, b \in S$, then $ab \in S$.

Given a multiplicative set, then one can construct a localized ring $S^{-1}R$ which I suppose the readers have known this. In order to be self-contained, I recall the construction:

In $R \times S$, we define an equivalent relation that $(r, s) \sim (r', s')$ if $(rs' - r's)t = 0$ for some $t \in S$. Let $\frac{r}{s}$ denote the equivalent class of $(r, s)$. One can define addition and multiplication naturally. The set of all equivalent classes, denoted $S^{-1}R$, is thus a ring. There is a natural ring homomorphism $\iota: R \rightarrow S^{-1}R$ by $\iota(r) = \frac{r}{1}$.

**Remark 0.2.**

(1) If $0 \in S$, then $S^{-1}R = 0$. We thus assume that $0 \not\in S$.

(2) If $R$ is a domain, then $\iota$ is injective. And in fact, $S^{-1}R \hookrightarrow F$ naturally, where $F$ is the quotient field of $R$.

(3) Let $J \lhd S^{-1}R$. We will use $J \cap R$ to denote the ideal $\iota^{-1}(J)$.
   
   (If $R$ is a domain, then $J \cap R = \iota^{-1}(J)$ by identifying $R$ as a subring of $S^{-1}R$).

I would like to recall the most important example and explain their geometrical meaning, which, I think, justify the notion of localization.

**Example 0.3.** Let $f \neq 0 \in k[x_1, ..., x_n]$ and let $S = \{1, f, f^2, ...\}$. The localization $S^{-1}k[x_1, ..., x_n]$ is usually denoted $k[x_1, ..., x_n]_f$. This ring can be regarded as "regular functions" on the open set $U_f := A^n_k - \mathcal{V}(f)$.
   
   One notices that $U_f$ is of course the maximal open subset that the ring $k[x_1, ..., x_n]_f$ gives well-defined functions.

**Example 0.4.** Let $x = (a_1, ..., a_n) \in A^n_k$ and $m_x = (x_1 - a_1, ..., x_n - a_k)$ be its maximal ideal. Take $S = k[x_1, ..., x_n] - m_x$, then the localization is denoted $k[x_1, ..., x_n]_{m_x}$. It is the ring of regular functions "near $x$".

Recall that for a $R$-module $M$, one can also define $S^{-1}M$ which is naturally an $S^{-1}R$-module. And we have:

**Proposition 0.5.**

(1) If $I \lhd R$, then $S^{-1}I \lhd S^{-1}R$. Moreover, every ideal $J \lhd S^{-1}R$ is of the form $S^{-1}I$ for some $I \lhd R$.

(2) For $J \lhd S^{-1}R$, then $S^{-1}(J \cap R) = J$.

(3) $S^{-1}I = S^{-1}R$ if and only if $I \cap S \neq \emptyset$.  

There is a one-to-one correspondence between \( \{ p \in \text{Spec}(R) | p \cap S = \emptyset \} \) and \( \{ q \in \text{Spec}(S^{-1}R) \} \).

In particular, the prime ideals of the local ring \( R_p \) are in one-to-one correspondence with the prime ideals of \( R \) contained in \( p \).

**Proof.**

1. If \( \frac{x}{s}, \frac{y}{t} \in S^{-1}I \), that is, \( x, y \in I \), then \( \frac{x}{s} + \frac{y}{t} = \frac{xt + ys}{st} \in S^{-1}I \). And if \( \frac{x}{s} \in S^{-1}R \) and \( \frac{y}{t} \in S^{-1}I \), then \( \frac{xt}{st} = \frac{ys}{st} \in S^{-1}I \). Hence \( S^{-1}I \) is an ideal.

   Moreover, let \( J \triangleleft S^{-1}R \). Let \( I := \iota^{-1}(J) \triangleleft R \). We claim that \( J = S^{-1}I \). To see this, for \( x \in I, \frac{x}{s} \in J \). Hence \( \frac{x}{s} = \frac{x}{s} \in J \) for all \( s \in S \). It follows that \( S^{-1}I \subset J \). Conversely, if \( \frac{x}{s} \in J \), then \( \frac{x}{s} = \frac{ys}{st} \in J \) and \( \frac{x}{s} = x(s) \). So \( \frac{x}{s} \in S^{-1}I \).

2. If \( \frac{x}{s} \in J \cap R \), then \( \frac{x}{s} = \frac{y}{t} \) for some \( y \in R \), i.e. \( (x - sy)t_0 = 0 \), for some \( t_0 \in S \). Then look at \( \frac{y}{t} \in S^{-1}(J \cap R) \). It’s clear that \( \frac{y}{t} \in S^{-1}(J \cap S) \).

   Conversely, if \( \frac{x}{s} \in J \), then \( \frac{x}{s} \in J \cap R \). And hence \( \frac{x}{s} \in S^{-1}(J \cap S) \).

3. If \( x \in I \cap S \), then \( \frac{1}{t} = \frac{x}{s} \in S^{-1}I \). Conversely, if \( \frac{1}{t} = \frac{x}{s} \in S^{-1}I \), then \( (x - s)t = 0 \). Therefore, \( xt = st \in S \cap I \).

4. For \( q \in \text{Spec}(S^{-1}R) \). It’s clear that \( q \cap R = \iota^{-1}q \in \text{Spec}(R) \).

   \( q \cap R \cap S = \emptyset \), otherwise \( q = S^{-1}(q \cap R) = S^{-1}R \) which is impossible.

   Conversely, let \( p \in \text{Spec}(R) \) and \( p \cap S = \emptyset \). We would like to show that \( S^{-1}p \) is a prime ideal. First of all, if \( \frac{1}{t} \in S^{-1}p \), then \( \frac{1}{t} = \frac{x}{s} \) for some \( x \in p \). It follows that \( (x - s)t = 0 \) for some \( t \in S \). Thus \( xt = st \in p \) which is a contradiction. Therefore, \( S^{-1}p \neq S^{-1}R \).

   Moreover, if \( \frac{x}{s} \) \( \in S^{-1}p \). Then \( \frac{yx}{st} = \frac{x}{t} \) for some \( x' \in p \) and \( s' \in S \). Then \( (ys' - stx')t' = 0 \) for some \( t' \in S \). Hence \( xys't \in p \). It follows that \( xy \in p \) since \( s't \notin p \). Thus either \( x \) or \( y \) in \( p \), and either \( \frac{x}{s} \) or \( \frac{y}{t} \) in \( S^{-1}p \).

   It remains to show that the correspondence is a one-to-one correspondence. We have seen that

\[ \iota^{-1} : \text{Spec}(S^{-1}R) \rightarrow \{ p \in \text{Spec}(R) | p \cap S = \emptyset \} \]

is surjective. By (2), it follows that this is injective.

5. Let \( S = R - p \), then \( S \cap q = \emptyset \) if and only \( q \subset p \).

\[ \square \]

But for \( I \triangleleft R \), then \( S^{-1}I \cap R \supset I \) only. Indeed, if \( x \in I \triangleleft R \). Then \( x = \frac{xt}{st} \in S^{-1}I \cap R \). Conversely, for \( x \in S^{-1}I \cap R \), then \( \frac{x}{s} = \frac{yt}{st} \) for some \( y \in I \). Thus \( (y - xt)s = 0 \) for some \( s, t \in S \). We cannot get \( y \in I \) in general. However, this is the case if \( I \) is prime and \( S \cap I = \emptyset \). Thus we have
Proposition 0.6. If \( p \triangleleft R \) is a prime ideal and \( S \cap p = \emptyset \). Then \( S^{-1}p \cap R = p \).

Corollary 0.7. If \( R \) is Noetherian (resp. Artinian), then so is \( S^{-1}R \).

Proof. This is an immediate consequence of (1) of the Proposition. \( \square \)

Example 0.8. Let \( p \in \text{Spec}(R) \), then \( R_p \) is a local ring with the unique maximal ideal \( pR_p \). To see that, if there is a maximal ideal \( m \). By the correspondence, \( m = qR_p \) for some \( q \subset p \). Thus \( m \subset pR_p \) and thus must be equal.

A ring with a unique maximal ideal is called a local ring. Thus \( R_p \) is a local ring.

Proposition 0.9. The operation \( S^{-1} \) commutes with formation of finite sums, product, intersection and radicals.

Proof. These can be checked directly. \( \square \)

Another important feature is that

Proposition 0.10. The operation \( S^{-1} \) is exact. That is, if \( M' \xrightarrow{f} M \xrightarrow{g} M'' \) is an exact sequence of \( R \)-module, then \( S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M'' \) is exact as \( S^{-1}R \)-module.

Proof. By the construction, it follows directly that \( S^{-1}g \circ S^{-1}f = 0 \). It suffices to check that \( \text{Ker}(S^{-1}g) \subset \text{Im}(S^{-1}f) \). If \( \frac{x}{s} \in \text{Ker}(S^{-1}g) \), then \( \frac{g(x)}{s} = 0 \in S^{-1}M'' \). That is \( tg(x) = 0 \) for some \( t \in S \). We have \( tg(x) = g(tx) = 0 \). And then \( tx = f(y) \) for some \( y \in M' \). Therefore,

\[
\frac{x}{s} = \frac{xt}{st} = \frac{f(y)}{st} = S^{-1}f\left(\frac{y}{st}\right).
\]

Corollary 0.11. The operation \( S^{-1} \) commutes with passing to quotients by ideals. That is, let \( I \triangleleft R \) be an ideal and \( \bar{S} \) the image of \( S \) in \( \bar{R} := R/I \). Then \( S^{-1}R/S^{-1}I \cong \bar{S}^{-1}\bar{R} \).

Proof. By considering \( 0 \rightarrow I \rightarrow R \rightarrow \bar{R} \rightarrow 0 \) as an exact sequence of \( R \)-modules. We have \( S^{-1}R/S^{-1}I \cong \bar{S}^{-1}\bar{R} \) as \( S^{-1}R \)-modules. We claim that there is a natural bijection from \( S^{-1}\bar{R} \) to \( \bar{S}^{-1}\bar{R} \) by \( \frac{x}{s} \mapsto \frac{x}{s} \) which is compatible with all structures.

One can also try to prove this directly. Basically, construct a surjective ring homomorphism \( S^{-1}R \rightarrow \bar{S}^{-1}\bar{R} \) and shows that the kernel is \( S^{-1}I \). We leave it as an exercise. \( \square \)