Advanced Algebra II
Homework 10
due on May. 21, 2004

$k$ is an algebraically closed field unless otherwise stated.

(1) Consider the ring homomorphism \( \varphi : k[x, y, z, w] \rightarrow k[s, t] \) by \( x \mapsto s^3, y \mapsto s^2t, z \mapsto st^2, w \mapsto t^3 \). Determine \( \ker \varphi \). Is \( \ker \varphi \) a prime ideal?

(2) Determine all prime ideal of \( k[x, y] \).

(3) Let \( R \) be a ring. And let \( \text{Spec}(R) \) be the set of all prime ideals of \( R \). For an ideal \( I \lhd R \), we define

\[ V(I) := \{ p \in \text{Spec}(R) | I \subset p \} \]

Show that we can define the "Zariski topology" on \( \text{Spec}(R) \) by considering \( V(I) \) as closed sets.

(4) Consider \( R := k[x, y]/(x^n - y^m) \), where \( (n, m) = 1 \). Show that \( (x^n - y^m) \) is prime. Find an algebraically independent element \( t \in R \) such that \( R \) is integral over \( k[t] \).

Moreover, let’s define the degree of the extension \( R/k[t] \) to be \( [K : k(t)] \), where \( K \) denote the quotient field of \( R \). What’s the degree of your extension \( R/k[t] \)? What’s the minimum possible degree?