Advanced Algebra I

Burnside’ Theorem

As an application, we are going to prove Burnside’s $p^aq^b$ theorem.

**Theorem 0.1.** Let $G$ be a group of order $p^aq^b$. Then $G$ is solvable.

**Proof.** If $G$ has a non-trivial normal subgroup $N \triangleleft G$. Then by induction on $(a, b)$, one sees that both $N$ and $G/N$ are solvable. Hence $G$ is solvable.

We thus assume that $G$ is non-abelian simple. Since $G$ is simple, each representation is either injective or trivial. We remark that if $\rho : G \to GL(V)$ is an irreducible representation of degree 1, then $\rho$ must be trivial cause $G$ can’t inject into an abelian group. Thus, any non-trivial irreducible representation is of degree $> 1$.

Let $\rho_1, \ldots, \rho_r$ be representative of isomorphic classes of irreducible representations with $\rho_1$ is the trivial representation. And Let $h_i$ be the number of elements in conjugacy classes $c_i$ (with $c_1$ being the class of the identity $e$, hence $h_1 = 1$). One has the following equations:

$$g = p^aq^b = 1 + \sum_{i=2}^{r} h_i,$$

$$0 = \chi_{\text{reg}}(s) = 1 + \sum_{i=2}^{r} d_i \chi_i(s),$$

for $s \neq e$.

Modulo the first equation by $q$, one finds that

$$h_i \not\equiv 0 \pmod{q}$$

for some $i \geq 2$. Recall that for any $s \in c_i$, one has that $h_i |C_G(s)| = |G| = p^aq^b$, it follows that $h_i = p^d$.

Next modulo the second equation by $p$, one finds that

$$d_i \chi_i(s) \not\equiv 0 \pmod{p}$$

for some $i \geq 2$. In particular, $p \nmid d_i$ and $\chi_i(s) \neq 0$.

Our goal is to prove the following

**Claim.** Under the condition that (for $i \geq 2$) $(d_i, h_i) = 1$, $\chi_i(s) \neq 0$ and $s \in c_i$. One has $\rho_i(s) = \lambda I$ for some $\lambda$.

Grant this for the time being. Then $\rho_i : G \to GL(V_i)$ gives an injection. $\rho_i(s) = \lambda I$ implies that $s \in Z(GL(V_i))$ and hence $s \in Z(\rho_i(G)) \cong Z(G)$. In particular, $G$ has non-trivial center, hence a non-trivial normal subgroup. This completes the proof.

**Proof.** It remains to prove the claim. Let $e_c = \sum_{t \in c_i} e_t \in \mathbb{C}[G]$, then $e_c \in Z(\mathbb{C}[G])$ and hence $\tilde{\rho}(e_c) = ll$ with $l = \frac{h_i}{d_i} \chi_i(s)$. More precisely, one has $\omega_i : Z(\mathbb{C}[G]) \to \mathbb{C}$ such that $\omega_i(e_c) = l$. Since $e_c \in Z(\mathbb{C}[G])$ is
integral over \( \mathbb{Z} \). It’s clear that \( l \) is integral over \( \mathbb{Z} \). Moreover, it’s clear that \( \chi_i(s) \) is integral over \( \mathbb{Z} \). Since \( (d_i, h_i) = 1 \) there exists \( x, y \in \mathbb{Z} \) such that \( d_i x + h_i y = 1 \). Thus one has

\[
\frac{1}{d_i} \chi_i(s) = x \chi_i(s) + \frac{h_i y}{d_i} \chi_i(s) \in \mathcal{A}.
\]

Now

\[
\chi_i(s) = tr(\rho_i(s)) = \lambda_1 + \ldots + \lambda_{d_i}
\]

is sum of root of unity. And \( 0 \neq \frac{\chi_i(s)}{d_i} \in \mathcal{A} \). We leave it as an exercise to show that

\[
\lambda_1 = \ldots = \lambda_{d_i} = \frac{1}{d_i} \chi_i(s).
\]

Therefore, \( \rho_i(s) = \lambda_1 I \). \hfill \Box

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