Advanced Algebra I

representation of finite groups, II Characters

Let \( \rho \) be a 1-dimensional representation of a group \( G \). Then in this case \( \rho = \chi : G \to \mathbb{C}^* \). One sees that \( \chi(st) = \chi(s)\chi(t) \) for all \( s, t \in G \). Such character is called an abelian character.

Let \( \hat{G} \) be the set of all 1-dimensional characters, it forms a group under the multiplication \( \chi\chi' : G \to \mathbb{C}^* \).

Exercise 0.1. Let \( G \) be an abelian group. Prove that \( G \cong \hat{G} \).

Recall that a representation \( \rho : G \to GL(V) \) is the same as a linear action \( G \times V \to V \). Suppose now that there are two representation \( \rho, \rho' \) on \( V, V' \) respectively. A linear transformation \( T : V \to V' \) is said to be \( G \)-invariant if it’s compatible with representations. That is,

\[
T\rho_s(v) = \rho'_s(Tv),
\]

for all \( v \in V \).

Thus an isomorphism of representation is nothing but a \( G \)-invariant bijective linear transformation.

Exercise 0.2. It’s easy to check that if \( T : V \to V' \) is \( G \)-invariant, then the \( \ker(T) \subset V \) and \( \text{im}(T) \subset V' \) are \( G \)-invariant subspaces.

Theorem 0.3 (Schur’s Lemma). Let \( \rho, \rho' \) be two irreducible representation of \( G \) on \( V, V' \) respectively. And let \( T : V \to V' \) be a \( G \)-invariant linear transformation. Then

1. Either \( T \) is an isomorphism or \( T = 0 \).
2. If \( V = V', \rho = \rho' \), then \( T \) is multiplication by a scalar.

Proof.

1. Since \( \ker(T) \) is a \( G \)-invariant subspace and \( V \) is irreducible. One has that either \( \ker(T) = 0 \) or \( \ker(T) = V \). Hence \( T \) is injective or \( T = 0 \). If \( T \) is injective, by looking at \( \text{im}(T) \), one must have \( \text{im}(T) = V' \). Therefore \( T \) is an isomorphism.

2. Let \( \lambda \) be an eigenvalue of \( T \). One sees that \( T_1 := T - \lambda I \) is also an \( G \)-invariant linear transformation. Since \( \ker(T_1) \) is non-zero, one has that \( \ker(T_1) = V \). Thus \( T_1 = 0 \) and hence \( T = \lambda I \).

\[ \square \]

Suppose one has \( T : V \to V' \) not necessarily \( G \)-invariant. One can produce an \( G \)-invariant linear transformation by the "averaging process". For \( T(v) = s^{-1}T(sv) \), we set

\[
\tilde{T}(v) := \frac{1}{g} \sum_{s \in G} s^{-1}T(sv).
\]

And it’s easy to check that this is \( G \)-invariant.
proof of the main theorem. (1) Let \( \rho, \rho' \) be two irreducible representation of \( G \) on \( V, V' \) with character \( \chi, \chi' \) respectively.

Let \( T : V \to V' \) be any linear transformation. One can produce a \( G \)-invariant transformation \( \tilde{T} \).

Suppose first that \( \rho \) and \( \rho' \) are not isomorphic. Then by Schur’s Lemma, \( \tilde{T} = 0 \) for all \( T \).

We fix bases of \( V, V' \) and write everything in terms of matrices.

\[
0 = (\tilde{T})_{ij} = \sum_{t,k,l} (R^t_{t-1})_{ik}(T)_{kl}(R_l)_{lj}.
\]

Take \( T = E_{ij} \), then one has

\[
0 = \sum_{t,k,l} (R^t_{t-1})_{ik}(E_{ij})_{kl}(R_l)_{lj} = \sum_t (R^t_{t-1})_{ii}(R_l)_{jj}.
\]

Hence

\[
< \chi', \chi > = \sum_{t,i,j} (R^t_{t-1})_{ii}(R_l)_{jj} = 0.
\]

Suppose now that \( \rho = \rho' \), \( \chi = \chi' \). The averaging process and Schur’s Lemma gives

\[
\lambda I = \tilde{T} = \frac{1}{g} \sum_t R(t)T R_t.
\]

One notice that \( \lambda d = tr(\tilde{T}) = tr(T) \).

Now we set \( T = E_{ii} \), then

\[
\frac{1}{d} = (\lambda I)_{ii} = \frac{1}{g} \sum_t (R^t_{t-1})_{ik}(E_{ii})_{kl}(R_l)_{li} = \sum_t (R^t_{t-1})_{ii}(R_l)_{ii}.
\]

It follows that

\[
< \chi, \chi > = \sum_t \sum_i (R^t_{t-1})_{ii}(R_l)_{ii} = \sum_i \frac{1}{d} = 1.
\]

(2) A class function \( f \) on a group \( G \) is a complex value function such that \( f(s) = f(t) \) if \( s \) and \( t \) are conjugate. The space \( C \) of class function is clearly a vector space of dimension \( r \), where \( r \) denotes the number of conjugacy classes of \( G \). We claim that the set of character of irreducible representation form a orthonormal basis of \( C \).

We remark that inner product can be defined on any class function.

Suppose now that \( \phi \) is a class function which is orthogonal to every \( \chi_i \). For any character \( \chi \) of an irreducible representation \( \rho \), we can produce a linear transformation by averaging process \( T := \frac{1}{d} \sum_t \phi(t) \rho_t \). It’s clear that \( tr(T) = < \phi, \chi >= 0 \). One sees that \( \tilde{T} : V \to V \) is \( G \)-invariant. By Schur’s Lemma, \( T = \lambda I \). But \( Tr(T) = 0 \). Thus \( T = 0 \) for any character \( \chi \).
We apply to the regular representation $\rho : G \to \mathbb{C}[G]$, 

$$
0 = T(e_1) = \frac{1}{g} \sum_t \overline{\phi(t)} \rho_t(e_1) = \frac{1}{g} \sum_t \overline{\phi(t)} e_t.
$$

Since $e_t$ forms a basis for $\mathbb{C}[G]$, it follows that $\phi(t) = 0$ for all $t \in G$ and hence $\phi = 0$.

(3) We may assume that there are $r$ irreducible representation. And suppose that the regular representation $\rho$ is decomposed into $n_1 \rho_1 \oplus \ldots \oplus n_r \rho_r$. One notice that $\rho(1) = g$ and $\rho(t) = 0$ for all $t \neq 1$. By direct computation,

$$
d_i = \langle \chi_{\rho}, \chi_i \rangle = n_i,
$$

$$
g = \langle \chi_{\rho}, \chi_{\rho} \rangle = \sum_i d_i^2.
$$

To prove that $d_i | g$ need some extra work on the group algebra $\mathbb{C}[G]$ which we will do later.

□