Advanced Algebra I  
Sep. 19,20, 2003 (Fri. Sat.)

1. Set Theory

We recall some set theory that will be frequently used in the sequel or that is not covered in the basic college course.

1.1. Zorn’s Lemma.

Definition 1.1. A set $S$ is said to be partially ordered if there is a relation $\leq$ such that

1. $x \leq x$
2. if $x \leq y$ and $y \leq x$, then $x = y$.
3. if $x \leq y$ and $y \leq z$ then $x \leq z$.

We usually call a partially ordered set to be a POSET.

Definition 1.2. A pair of elements is said to be comparable if either $x \leq y$ or $y \leq x$. A set is said to be totally ordered if every pair is comparable.

We also need the following definition:

Definition 1.3. A maximal element of an poset $S$ is an element $m \in S$ such that if $m \leq x$ then $m = x$.

For a given subset $T \subset S$, an upper bound of $T$ is an element $b \in S$ such that $x \leq b$ for all $x \in T$.

One has

Theorem 1.4 (Zorn’s lemma). Let $S$ be a non-empty poset. If every non-empty totally ordered subset (usually called a ”chain”) has an upper bound, then there exists a maximal element in $S$.

Example 1.5. Let $R$ be a $\neq 0$ commutative ring. One can prove that there exists a maximal ideal by using Zorn’s lemma. The proof goes as following: Let $S = \{I \triangleleft R | I \neq R\}$ equipped with the $\subset$ as the partial ordering. $S \neq \emptyset$ because $0 \in S$. For a chain $\{I_j\}_{j \in J}$, one has a upper bound $I = \cup I_j$. Then we are done by Zorn’s lemma.

1.2. cardinality. In order to compare the ”size of sets”, we introduce the cardinality.

Definition 1.6. Two sets $A, B$ are said to have the same cardinality if there is a bijection between them, denoted $|A| = |B|$. And we said $|A| \leq |B|$ if there is a injection from $A$ to $B$.

It’s easy to see that the cardinality has the properties that $|A| \leq |A|$ and if $|A| \leq |B|, |B| \leq |C|$, then $|A| \leq |C|$. So it’s likely that the ”cardinality are partially ordered” or even totally ordered.
Lemma 1.7. Given two set $A, B$, either $|A| \leq |B|$ or $|B| \leq |A|$.

Sketch. Consider 

$$S = \{(C, f)|C \subset A, f : C \to B \text{ is an injection}\}.$$ 

Apply Zorn’s lemma to $S$, one has an maximal element $(D, g)$, then one claim that either $D = A$ or $\text{im}(g) = B$. 

Theorem 1.8 (Schroeder-Bernstein). If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Sketch. Let $f, g$ be the injection from $A, B$ to $B, A$ respectively. One needs to construct a bijection by using $f$ and $g$. Some parts of $A$ use $f$ and some parts not. So we consider the partition 

$$A_1 := \{a \in A|\text{has parentless ancestor in } A\},$$
$$A_2 := \{a \in A|\text{has parentless ancestor in } B\},$$
$$A_3 := \{a \in A|\text{has infinite ancestor}\}.$$ 

And so does $B$.

Then we claim that $f$ restricted to $A_1, A_3$ are bijections to $B_1, B_3$. And $g$ restricted to $B_2, B_3$ are bijections to $A_2, A_3$. So the desired bijection can be constructed.

We need some more properties of cardinality. If $|A| = |\{1, \ldots, n\}|$, then we write $|A| = n$. And if $|A| = |\mathbb{N}|$ then we write $|A| = \aleph_0$.

Proposition 1.9. If $A$ is infinite, then $\aleph_0 \leq |A|$.

Proof. By Axiom of Choice.

Definition 1.10. 

$$|A| + |B| := |A \sqcup B|,$$
$$|A| \cdot |B| := |A \times B|.$$ 

We have the following properties:

Proposition 1.11. 

(1) If $|A|$ is infinite and $|B|$ is finite, then $|A + B| = |A|$.

(2) If $|B| \leq |A|$ and $|A|$ is infinite, then $|A + B| = |A|$.

(3) If $|B| \leq |A|$ and $|A|$ is infinite, then $|A \times B| = |A|$.

Proof. For (1), take an countable subset $A_0$ in $A$, one sees that $|A_0| = |A_0| + |B|$ by shifting. Then we are done.

For (2), It’s enough to see that $|A + A| \leq |A|$. Pick an maximal subset $X \subset A$ having the property that $|X + X| \leq |X|$ (by Zorn’s Lemma). One claim that $A - X$ is finite, and then we are done by (1).

For (3), we leave it as an exercise to the readers.