Part A.

(1) Suppose that there are two representation $\rho, \rho'$ on $V, V'$ respectively. A linear transformation $T : V \rightarrow V'$ is said to be $G$-invariant if it’s compatible with representations. That is,

$$T\rho_s(v) = \rho'_s(Tv),$$

for all $v \in V$. Prove that if $T : V \rightarrow V'$ is $G$-invariant, then the $\ker(T) \subset V$ and $\text{im}(T) \subset V'$ are $G$-invariant subspaces.

(2) A class function $\chi$ is called an abelian character if $\chi(st) = \chi(s)\chi(t)$. Let $\hat{G}$ be the set of abelian characters. Show that $\hat{G}$ is naturally a group. And prove that if $G$ is a finite abelian group, then $|G| = |\hat{G}|$.

(3) Let $\rho$ be a representation of $G$ on $V$. Prove or disprove: If the only $G$-invariant linear transformation on $V$ are multiplication by scalar, then $\rho$ is irreducible.

Part B.

(1) Let $G$ be a non-abelian group of order 27 such that maximal order is 3.

(a) Show that the center of $G$ is a group of order 3.
(b) How many conjugacy classes are there in $G$?
(c) Determine the character table of $G$.
(d) Find an irreducible representation of degree 3.