Advanced Algebra I
Homework 12
due on Dec. 19, 2003

(1) Determine the Galois group of the following extension:
   (a) \( \mathbb{Q}(\sqrt{3}, \sqrt{5}) \) over \( \mathbb{Q} \).
   (b) Let \( F \) be the splitting field of \( x^5 - 2 \) over \( \mathbb{Q} \). What is the
       Galois group \( \text{Gal}_{F/\mathbb{Q}} \)?

(2) Show that if \([F : K] = 2\) then \( F \) is normal over \( K \).

(3) Prove that an algebraically closed field is infinite.

(4) Let \( K = \mathbb{K} \) be an algebraically closed field. Let \( f(x_1, \ldots, x_n) \neq 0 \in \mathbb{K}[x_1, \ldots, x_n] \). Prove that there are \( a_1, \ldots, a_n \in K \) such that
    \( f(a_1, \ldots, a_n) = 0 \). That is \( f = 0 \) has a solution in \( K^n \). What if
    \( K \) is not algebraically closed?

(5) Let \( F \) be a finite field of \( p^n \) elements and \( P \) be its prime field, that is, a subfield of \( p \) elements.
   (a) Consider \( \sigma : F \rightarrow F \) by \( \sigma(u) = u^p \). Show that \( \sigma \in \text{Gal}_{F/P} \).
   (b) Determine \( \text{Gal}_{F/P} \).