(1) Consider the field \( \mathbb{Q}[u] := \mathbb{Q}[x]/(x^3 + x + 1) \), where \( u \) denote the coset of \( x \).
   (a) Find \( u^{-2} \) in \( \mathbb{Q}[u] \).
   (b) Find minimal polynomial of \( u^2 \).

(2) Let \( F/K \) be a field extension. Let \( \mathcal{A} \subset F \) be those elements in \( F \) which is algebraic over \( K \). Show that \( \mathcal{A} \) is a field.

(3) In the group \( \text{GL}(2, \mathbb{F}_q) \), there is the Borel group \( B \) of upper triangular matrices and diagonal subgroup \( D \). There is a group homomorphism \( B \to D \) by \( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \). Fix \( \alpha, \beta : \mathbb{F}_q^* \to \mathbb{C}^* \), one has \( D \to \mathbb{F}_q^* \times \mathbb{F}_q^* \xrightarrow{(\alpha, \beta)} \mathbb{C}^* \times \mathbb{C}^* \xrightarrow{\text{mult}} \mathbb{C}^* \).

   It follows that the composition \( B \to \mathbb{C}^* \) is a representation. Let \( W_{\alpha, \beta} \) be the induced representation on \( \text{GL}(2, \mathbb{F}_q) \). Compute the character of \( W_{\alpha, \beta} \) and determine the irreducibility.

   Recall that conjugacy classes of \( \text{GL}(2, \mathbb{F}_q) \) are represented by elements of the following four types:

   \[
   a_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \quad b_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, \quad c_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad d_{x,y} = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}
   \]

(4) Let \( F/K \) be a field extension of degree 2 and \( \text{char}(K) \neq 2 \). Show that there is an element \( u \in F - K \) such that \( u^2 \in K \).

(5) Prove or disprove: Let \( F/K \) be a field extension of degree \( n \) and \( d \mid n \). Then there is an intermediate field \( E, K \subset E \subset F \), such that \( [E : K] = d \).