About Skew Partitions in Minimal Imperfect Graphs

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V. Chvátal conjectured in 1985 that a minimal imperfect graph $G$ cannot have a skew cutset (i.e., a cutset $S$ decomposable into disjoint sets $A$ and $B$ joined by all possible edges). We prove here the conjecture in the particular case where at least one of $A$ and $B$ is a stable set.

1. INTRODUCTION

For terms not defined here, the reader is referred to [6]. For a graph $G = (V, E)$, we denote by $\overline{G}$ the complement of $G$. We use $\omega(G)$ to denote the size of a largest clique in $G$ and $\pi(G)$ to denote the size of a largest stable set in $G$, or simply $\omega$ and $\pi$ when no confusion is possible.

A graph $G$ is said to be perfect if, for each induced subgraph $H$ of $G$, $H$ can be coloured with $\omega(H)$ colours such that every two adjacent vertices have different colours. A graph is minimal imperfect if it is not perfect but all of its proper induced subgraphs are perfect. One can easily check that an odd chordless cycle with at least five vertices (usually called an odd hole) and its complement (usually called an odd antihole) are minimal imperfect graphs. Berge [1] conjectured that the only minimal imperfect graphs are the odd holes and the odd antiholes. This conjecture is still unsettled and known as the strong perfect graph conjecture (SPGC). Berge also conjectured that a graph is perfect if and only if its complement is perfect. This conjecture is a consequence of the following theorem of Lovász [9]:

**Theorem 1.1 (The Perfect Graph Theorem).** A graph $G = (V, E)$ is perfect if and only if for every induced subgraph $H$ of $G$ the following inequality holds:

$$\pi(H) \cdot \omega(H) \geq |H|.$$

This theorem was the first step towards a new approach of minimal imperfect graphs. The results of Padberg [11], Tucker [12], and Lovász [9]...
provide a well-known list of properties satisfied by each minimal imperfect graph \( G \). We give three of them, which will be used later.

(\text{P1}) For every vertex \( v \in V \), the graph obtained from \( G \) by deleting the vertex \( v \), denoted by \( G - v \), has a unique partition into stable sets of size \( \alpha \) (\( \alpha \)-stables) and a unique partition into cliques of size \( \omega \) (\( \omega \)-cliques).

(\text{P2}) For every \( \alpha \)-stable \( S \) of \( G \), there exists precisely one \( \omega \)-clique of \( G \), denoted by \( K(S) \), such that \( S \cap K(S) = \emptyset \). For every \( \omega \)-clique \( K \) of \( G \), there exists precisely one \( \alpha \)-stable of \( G \), denoted by \( S(K) \), such that \( K \cap S(K) = \emptyset \).

(\text{P3}) For every \( \alpha \)-stable \( S \) of \( G \) and for every vertex \( v \in V \), \( v \) belongs to \( S \) if and only if \( K(S) \) is an element of the partition of \( G - v \) into \( \omega \)-cliques; for every \( \omega \)-clique \( Q \) of \( G \) and for every vertex \( v \in V \), \( v \) belongs to \( K \) if and only if \( S(K) \) is an element of the partition of \( G - v \) into \( \alpha \)-stables.

Many people believe that an important ingredient in the proof of the SPGC will be the resolution of the following conjecture due to Chvátal [2]. First, we recall some definitions: the partition \( \{A,B,C,D\} \) of \( V \) into four nonempty sets is said to be a skew partition of \( G \) if there are all possible edges between \( A \) and \( B \) and no edges between \( C \) and \( D \). Such a skew partition is denoted by \( (A,B,C,D) \). The set \( A \cup B \) is called a skew cutset. We remark that \( (A,B,C,D) \) is a skew partition of \( G \) if and only if \( (C,D,A,B) \) is a skew partition of \( G \).

**Conjecture 1.1.** No minimal imperfect graph admits a skew partition.

In [8], de Figueiredo et al. present a polynomial-time algorithm for testing whether a graph admits a skew partition.

Chvátal’s motivation for making the previous conjecture was the following result known as the star cutset lemma. A set \( X \) of vertices of a connected graph \( G \) is called a star cutset if the graph obtained from \( G \) by deleting the vertices of \( X \) is not connected and there exists a vertex \( x \) in \( X \) which is adjacent to all the other vertices of \( X \).

**Lemma 1.1 (Chvátal [2]).** No minimal imperfect graph contains a star cutset.

Other cases of Conjecture 1.1 have been proved by Cornuéjols and Reed [3] as well as Hoàng [7]:

**Lemma 1.2 (Cornuéjols, Reed [3]).** No minimal imperfect graph contains a skew cutset which induces a multi-partite graph.
In [7] Hoa-ng defined two particular types of skew cutsets and proved that no minimal imperfect graph contains such skew cutsets. Let \( G \) be a graph with a skew partition \((A, B, C, D)\). The set \( A \cup B \) is a \( U \)-cutset if there are distinct vertices \( u_1, u_2 \in C \) such that \( u_1 \) is adjacent to all the vertices of \( A \) and \( u_2 \) is adjacent to all the vertices of \( B \). The set \( A \cup B \) is a \( T \)-cutset if there are vertices \( u_1 \in C, u_2 \in D \) such that each of the vertices \( u_1 \) and \( u_2 \) is adjacent to all the vertices of \( A \). Hoa-ng proved the two following results:

**Lemma 1.3.** No minimal imperfect graph contains a \( U \)-cutset.

**Lemma 1.4.** No minimal imperfect graph contains a \( T \)-cutset.

In this paper, we prove the following theorem which generalizes Lemma 1.2:

**Theorem 1.2.** No minimal imperfect graph admits a skew partition \((A, B, C, D)\) such that \( A \) is a stable set.

Let us recall the usual notations. For any chordless path \( P \), the length of \( P \) is the number of its edges. If \( V(P) = \{v_1, \ldots, v_k\} \) and \( E(P) = \{v_i v_{i+1} \mid i \in \{1, \ldots, k-1\}\} \), \( P \) is denoted by \([v_1, \ldots, v_k]\). The vertices \( v_1 \) and \( v_k \) are its end-vertices, while every vertex \( v_i \), with \( 1 < i < k \), is said to be an internal vertex. The neighbourhood of a vertex \( x \) in \( G \) is denoted by \( N_G(x) \) or simply \( N(x) \) when no confusion is possible. The subgraph of \( G \) induced by a part \( X \) of \( V \) is denoted by \( G[X] \). If \( W \) is a proper induced subgraph of \( G \), we denote by \( W + x \) the subgraph \( G[W] \cup \{x\} \).

The proofs explained in the next section allow us to say that Chvátal’s conjecture is true if the following statement is true: any minimal skew cutset in a minimal imperfect graph contains three vertices \( u, v, w \) such that there exists an odd chordless path \([u, x_1, \ldots, x_{2k}, v]\) from \( u \) to \( v \), with no vertex in \( N(w) \) and no internal vertex in the cutset.

### 2. MAIN RESULTS

We now present the proof of the main theorem. First, we give a famous result proved by Fonlupt and Uhry and independently by Meyniel. We need to recall a definition: two nonadjacent vertices \( x \) and \( y \) form an even pair if all of the chordless paths from \( x \) to \( y \) have an even number of edges.

**Lemma 2.1** (Fonlupt and Uhry [4], Meyniel [10]). No minimal imperfect graph contains an even pair.
We shall also need two technical lemmas which will be proved in the last section of the paper. One of these is a particular case of Theorem 1.2. Before stating this lemma, we have to give a definition. Let \((A, B, C, D)\) be a skew partition of a graph \(G\). If \(|A| = 2\) or \(|B| = 2\), \(A \cup B\) is called a double star cutset.

**Lemma 2.2.** No minimal imperfect graph admits a double star cutset.

Finally, the following notion and lemma make up the core of the proof of Theorem 1.2. We say that a graph \(G = (V, E)\) is a \(T\)-graph if \(G\) is a Berge graph (i.e., a graph which contains no odd hole and no odd antihole) whose vertex set admits a partition \(V = \{u, v\} \cup X \cup Y (X \neq \emptyset, Y \neq \emptyset)\) such that:

- \(uv \notin E; X \subseteq N(u) \cap N(v);\)
- \(\overline{G}[X]\) is connected;
- \(\{u, v\} \cup Y\) induces an odd chordless path between \(u\) and \(v\).

Such a graph is denoted by \(T(u, v, X, Y)\). An example of \(T\)-graph is depicted in Fig. 1.

**Lemma 2.3.** Let \(G = T(u, v, X, Y)\) be a \(T\)-graph. If \(G\) satisfies the conditions:

(i) for each pair of adjacent vertices \(y, y' \in Y\), there exists no odd chordless path between \(y\) and \(y' \in \overline{G}[\{y, y'\} \cup X];\)

(ii) for each pair of nonadjacent vertices \(x, x' \in X\) there exists no odd chordless path between \(x\) and \(x' \in \overline{G}[\{x, x'\} \cup Y],\)

then there exists a vertex in \(Y\) which is adjacent to all the vertices of \(X\).

**Remark 2.1.** Let \(G = T(u, v, X, Y)\) be a \(T\)-graph. If the path \(\overline{G}[\{u, v\} \cup Y]\) has a length greater than three, then the condition (i) holds; otherwise, we can find an odd antihole in \(G\).

We shall also use this lemma in the proof of Lemma 2.2. We can make the following remark about Berge graphs:

![T-graph](image)
Remark 2.2. Let $G$ be a Berge graph, let $S$ be a cutset of $G$, and let $C$ be a connected component of $G[V\setminus S]$. Let $D$ be the set $V\setminus (S \cup C)$. If $u$ and $v$ are two nonadjacent vertices in $G[S]$ such that there exist chordless paths between $u$ and $v$ in both of the graphs $G[u,v] \cup C$ and $G[u,v] \cup D$, then all these paths have the same parity; otherwise, we can find an odd hole in $G$.

We can now begin the proof of the main theorem.

*Proof of Theorem 1.2.* Let $G=(V,E)$ be a minimal imperfect graph. Clearly, Theorem 1.2 is true for odd holes and odd antiholes. So, we can consider that $G$ is a Berge graph. To derive a contradiction, we assume that $G$ admits a skew partition $(A,B,C,D)$ such that $A$ is a stable set. By Lemmas 1.1 and 2.2, we can suppose that $|A| \geq 3$, and, without loss of generality, we can also suppose that $A \cup B$ is a minimal cutset of $G$. So, each vertex of $A$ has neighbours in each of the connected components of $G[C \cup D]$. Then, by Remark 2.2, for every pair of vertices $a, b \in A$, all the chordless paths between $a$ and $b$ in $G[a,b] \cup C \cup D$ have the same parity. We have:

**Claim 2.1.** There exist three vertices $u, v, w \in A$ and an odd chordless path $[y_1, ..., y_2l]$ in $G[C \cup D]$ such that $[u, y_1, ..., y_2l, v]$ is a chordless path from $u$ to $v$ and $w$ is adjacent to none of the vertices $y_1, ..., y_2l$.

**Proof.** We distinguish two cases:

**Case 1.** There exist two vertices $a, b \in A$ such that each chordless path from $a$ to $b$ in $G[a,b] \cup C \cup D$ has an even length.

Since $\{a, b\}$ is not an even pair in $G$ (by Lemma 2.1), there exists an odd chordless path, $P=[a, z_1, ..., z_{2k}, b]$, from $a$ to $b$ in $G$. Since every vertex of $B$ is adjacent to both $a$ and $b$, no vertex of $\{z_1, ..., z_{2k}\}$ belongs to $B$. Then we have $\{z_1, ..., z_{2k}\} \subseteq A \cup C \cup D$. Now, since $P$ has an odd length and $A$ is a stable set, we can easily see that there exists an odd chordless subpath of $P$, denoted by $[u, y_1, ..., y_{2l}, v]$, such that $u, v \in A$ and $\{y_1, ..., y_{2l}\} \subseteq C \cup D$. Clearly, one of the vertices $a$ and $b$, denoted by $w$, is distinct from $u$ and distinct from $v$. So $w$ is adjacent to none of the vertices $y_1, ..., y_{2l}$.

**Case 2.** For every pair of vertices $a, b \in A$, every chordless path joining $a$ to $b$ in $G[a,b] \cup C \cup D$ has an odd length.

Let us choose an odd chordless path $P=[u, y_1, ..., y_{2l}, v]$ in $G[A \cup C \cup D]$ such that $u, v \in A$ and there exists no shorter chordless path between any two vertices of $A$ in $G[A \cup C \cup D]$. Then, we can remark that no vertex of $\{y_1, ..., y_{2l}\}$ belongs to $A$. Let $w$ be a vertex of $A$, distinct from $u$ and
distinct from $v$. So, clearly, $w$ has no neighbour in \{ $y_1, ..., y_{2l}$ \}; otherwise, $P$ is not minimum. 

Let us choose vertices $u, v, w, y_1, ..., y_{2l}$ as in Claim 2.1. We can suppose, without loss of generality, that vertices $y_1, ..., y_{2l}$ belong to $C$. Let $Q$ be a connected component of $G[B]$. We claim:

**Claim 2.2.** The $T$-graph $T(u,v,Q,\{y_1, ..., y_{2l}\})$ satisfies conditions (i) and (ii) of Lemma 2.3.

**Proof.** Suppose that the condition (i) is unsatisfied. By Remark 2.1, we have $l=1$. Moreover, there exists $\{ y_1, q_1, ..., q_{2k}, y_2 \}$, an odd chordless path between $y_1$ and $y_2$ in $G[\{ y_1, y_2 \} \cup Q]$. Therefore, the graph $G[\{ y_1, q_1, ..., q_{2k}, y_2, w \}]$ is an odd antihole, a contradiction.

Suppose that the condition (ii) is unsatisfied. Let $q_1, q_2$ be nonadjacent vertices of $Q$ and let $y_1, ..., y_2$ be vertices in $\{ y_1, y_{2l} \}$ such that $G[\{ q_1, y_1, ..., y_2, q_2 \}]$ is an odd chordless path from $q_1$ to $q_2$. Therefore, by Claim 2.1, the graph $G[\{ w, q_1, y_1, ..., y_{2l}, q_2 \}]$ is an odd hole, a contradiction.

Now, by Lemma 2.3, there exists in $\{ y_1, ..., y_{2l} \}$, a vertex $y$ which is adjacent to all the vertices of $Q$. Let $\{ y, d_1, ..., d_{2p}, v \}$ be an odd chordless path from $u$ to $v$ in $G[\{ u, v \} \cup D]$. The $T$-graph $T(u,v,Q,\{d_1, ..., d_{2p}\})$ satisfies conditions (i) and (ii) of Lemma 2.3; otherwise, one can easily see that there exists an odd hole or an odd antihole in $G[\{ y \} \cup Q \cup \{ d_1, ..., d_{2p} \}]$. Then, by Lemma 2.3, there exists, in $\{d_1, ..., d_{2p}\}$, a vertex $d$ which is adjacent to all of the vertices of $Q$. Hence, $Q \cup (A \cup B) \setminus Q$ is a $T$-cutset of $G$, which is in contradiction with Lemma 1.4. So, Theorem 1.2 is proved.

3. PROOFS OF TECHNICAL LEMMAS

**Proof of Lemma 2.3.** First, we remark:

**Remark 3.1.** Let $G=(V,E)$ be a graph. If $V$ is partitionable into two sets $\{ V_1, V_2 \}$ such that $V_1$ is a stable set and $V_2$ induces a chordless path of $G$, then $G$ contains no odd antihole $C_{2k+1}$ with $l \geq 3$.

Assume the contrary. Let $\bar{C}=\bar{C}_{2k+1}$ (with $k \geq 3$) be an odd antihole of $G$. Since $\alpha(C)=2$, we have $|V(\bar{C}) \cap V_1| \leq 2$ (because $G[V_1]$ is a stable). Hence, the set $H=V(\bar{C}) \cap V_2$ contains at least five vertices. Now, since $G[V_2]$ is a chordless path, it is easy to see that $\alpha(G[H]) \geq 3$, which is incompatible with the fact that $G[H]$ is an induced subgraph of $C$.

We now prove a particular case of Lemma 2.3.
Claim 3.1. Let \( G \) be a T-graph \( T(u, v, X, Y) \) such that \( X \) is a stable set. Then Lemma 2.3 is true for \( G \).

**Proof.** Let \( G = T(u, v, X, Y) \) be a T-graph satisfying the conditions (i) and (ii) of Lemma 2.3 and such that \( X \) is a stable set. Recall that \( G \) contains no odd hole and no odd antihole.

We can remark that if \( X \) has only one vertex, noted \( x \), then there exists a vertex \( y \in E \) such that \( xy \in E \) (otherwise \( G[\{x, y, v\} \cup Y] \) is an odd hole) and so Lemma 2.3 is true.

Now, the proof is by induction on the number of vertices of \( X \cup Y \). Clearly, we have \( |X \cup Y| \geq 3 \). If \( |X \cup Y| = 3 \), then \( X \) contains only one vertex, and, by the previous remark, we are done. Assume that the claim is true for T-graphs \( T(u, v, X, Y) \) such that \( X \) is a stable set, \( |X \cup Y| < q \), and \( q \geq 4 \).

Let \( G \) be a T-graph \( T(u, v, X, Y) \) satisfying conditions (i) and (ii), such that \( X \) is a stable set and \( |X \cup Y| = q \).

We denote by \( y_1, y_2, y_3, \ldots, y_{2k} (k \geq 1) \) the vertices of \( Y \) such that \( \{u, y_1, \ldots, y_{2k}, v\} \) is a path from \( u \) to \( v \). We set \( X = \{x_1, \ldots, x_l\} \). We have seen that if \( |X| = 1 \), then Lemma 2.3 is true. Then, we assume that \( X \) contains at least two vertices.

For every \( i \in \{1, \ldots, l\} \), we note \( Y_i \) as the set of all the vertices \( y \in Y \) such that \( N(y) \supseteq X \setminus \{x_i\} \). We can observe that each T-graph \( H_i = T(u, v, X \setminus \{x_i\}, Y) \) verifies conditions (i) and (ii). Moreover, \( X \setminus \{x_i\} \) is a stable set of \( H_i \). Hence, applying the induction hypothesis, we obtain that there exists a vertex \( y \in Y \) adjacent to all the vertices of \( X \setminus \{x_i\} \). Therefore, for each \( i \in \{1, \ldots, l\} \), \( Y_i \) is a nonempty set. If there exist \( i \neq j \) (with \( i, j \in \{1, \ldots, l\} \)) such that \( Y_i \cap Y_j \neq \emptyset \), then the proof is finished (a vertex \( y \in Y_i \cap Y_j \) is adjacent to every vertex of \( X \)). Hence, in the following, we suppose that for each \( i, j \in \{1, \ldots, l\} \) such that \( i \neq j \), we have \( Y_i \cap Y_j = \emptyset \).

If \( |Y| = 2 \) then \( X \) cannot have more than two vertices \( x_1, x_2 \) (otherwise we would have \( i \neq j \) such that \( Y_i \cap Y_j \neq \emptyset \)). Then we can suppose, without loss of generality, that \( y_1 \in Y_1 \) and \( y_2 \in Y_2 \). Thus \( \{x_1, y_2, y_1, x_2\} \) is an odd chordless path from \( x_1 \) to \( x_2 \) in \( G[\{x_1, x_2\} \cup Y] \), which is in contradiction with the fact that \( G \) verifies the condition (ii). Hence, we have \( |Y| \geq 4 \).

Now, let us show that at least one vertex of \( \{y_2, \ldots, y_{2k-1}\} \) belongs to \( \bigcup_{i \in \{1, \ldots, l\}} Y_i \). In the contrary case, since \( X \) contains at least two vertices, we can suppose, without loss of generality, that \( Y_1 = \{y_1\} \) and \( Y_2 = \{y_{2k}\} \). Moreover, we have \( |X| = 2 \) (otherwise, a vertex of \( Y_3 \) belongs to \( \{y_2, \ldots, y_{2k-1}\} \), a contradiction). So \( x_1 \) (resp. \( x_2 \)) is adjacent to no vertex of \( \{y_2, \ldots, y_{2k-1}\} \). Then \( \{x_2, y_1, \ldots, y_{2k}, x_1\} \) is an odd chordless path between \( x_1 \) and \( x_2 \) in \( G[\{x_1, x_2\} \cup Y] \), which is in contradiction with the condition (ii). Therefore, there exists a vertex \( y_i \in \{y_2, \ldots, y_{2k-1}\} \) such that \( y_i \) belongs to a set \( Y_j \) (with \( i \in \{1, \ldots, l\} \)). To simplify the notation, we suppose that \( Y_i = Y_1 \).
We have $N(y_j) \cap X = X \setminus \{x_1\}$. Since $[y_1, \ldots, y_{2k}]$ is an odd path, one of the two paths $[y_1, \ldots, y_{2k-1}, y_1]$ or $[y_{r+1}, \ldots, y_{2k}]$ has an odd length. We suppose, without loss of generality, that $[y_1, \ldots, y_s]$ is an odd path.

Let $y_r$ be a neighbour of $x_1$ on the chordless path $[y_0 = u, y_1, \ldots, y_r]$, such that $y_r$ is the closest to $y_s$ (since $x_1u \in E$, $y_s$ exists). Let $y_{r'}$ be a neighbour of $x_1$ in the chordless path $[y_r, \ldots, y_{2k}, y_{2k+1} = v]$, such that $y_{r'}$ is the closest to $y_s$ (since $x_1v \in E$, $y_s$ exists). Since $x_1$ is not adjacent to $y_s$, we have $y_r \neq y_s$ and $y_{r'} \neq y_s$. Clearly, $[x_1, y_r, y_{r+1}, \ldots, y_{2k-1}, y_{r'}]_X$ is a chordless cycle of $G$. Since $G$ contains no odd hole, the chordless paths $[y_1, \ldots, y_r]$ and $[y_{r'}, \ldots, y_{2k-1}, y_1]$ have lengths of same parity. So, we obtain the two following cases:

**Case 1.** The chordless paths $[y_r, \ldots, y_s]$ and $[y_{r'}, \ldots, y_{2k-1}, y_1]$ are odd.

We consider the graph $G_1 = G[\{u, y_1, \ldots, y_r\} \cup \{v\} \cup X] + y_s v$ (i.e., the graph obtained from $G[\{u, y_1, \ldots, y_r\} \cup \{v\} \cup X]$ by adding the edge $y_s v$). Clearly, in $G_1$, $[u, y_1, \ldots, y_r, v]$ is an odd chordless path. Let us prove that $G_1$ is a Berge graph.

By Remark 3.1, $G_1$ contains no odd antihole with more than five vertices. Thus, we only need to prove that $G_1$ contains no odd hole. Assume, to derive a contradiction, that there exists an odd hole, noted $C$; in $G_1$. Since $C$ cannot be an odd hole in $G$, $C$ contains the edge $y_s v$. We set $C' = [y_r, v, a_1, \ldots, a_{2r-1}, y_s]$ (with $r \geq 1$). Since, in $G_1$, $x_1$ is the only vertex which is both adjacent to $v$ and nonadjacent to $y_s$, we have $a_1 = x_1$. Moreover, the fact that $v$ is adjacent to every vertex of $X$ implies that $[a_2, \ldots, a_{2r-1}, y_s]$ is a chordless subpath of $[u, y_1, \ldots, y_r]$.

Since $x_1$ is nonadjacent to each vertex of $[a_3, \ldots, a_{2r-1}, y_s]$, we have $a_2 = y_s$. Then we have $C = [y_r, v, x_1, y_s, y_{r+1}, \ldots, y_{2k-1}, y_1]_Y$, which is in contradiction with the fact that $C$ is an odd hole. Therefore, $G_1$ is a Berge graph.

Now, $G_2 = G[\{y_r, \ldots, y_{2k}, y_s\} \cup X] + x_1 y_r$. Recall that $[y_r, \ldots, y_{2k}, v]_X$ is an odd chordless path and remark that, in $G_2$, $y_r$ is adjacent to all the vertices of $X$. Let us prove that $G_2$ is a Berge graph.

By Remark 3.1, we only need to prove that $G_2$ contains no odd hole. Assume, to derive a contradiction, that there exists an odd hole $C$ in $G_2$. Since $C$ is not an odd hole in $G$, $C$ contains the edge $x_1 y_r$. We set $C = [x_1, y_r, a_1, \ldots, a_{2r-1}, x_1]$ (with $r \geq 1$). If $a_1$ is a vertex of $X$, then we can
observe that \([a_2, \ldots, a_{2k+1}]\) is an induced path from \(G[\{y_{1,2}, \ldots, y_{2k}\}]\) (because \(y_i\) is adjacent to every vertex of \(X\) and \(v\) is adjacent to both \(x_1, a_1\)). Then, \([a_2, a_3, \ldots, a_{2k+1}, x_1]\) is an odd chordless path between \(a_1\) and \(x_1\) in \(G[\{a_1, x_1\} \cup Y]\), which is in contradiction with the fact that \(G\) verifies the condition (ii). So, we have \(a_1 = y_{i-1}\). It is not difficult to see that \(C = \{x_1, y_i, y_{i+1}, \ldots, y_{k}, x_k\}\), which is incompatible with the fact that \(C\) is an odd hole. Therefore \(G_2\) is a Berge graph.

Hence, \(G_2 = T(y_i, v, X, \{y_{i+1}, \ldots, y_{2k}\})\) is a \(T\)-graph and we can remark that \(G_2\) verifies conditions (i) and (ii) of Lemma 2.3. We apply the induction hypothesis to deduce that there exists, in \(G_2\), a vertex \(y \in \{y_{i+1}, \ldots, y_{2k}\}\) which is adjacent to every vertex of \(X\). Then, in \(G\), \(y\) is adjacent to every vertex of \(X\), a contradiction.

Using Claim 3.1, we are going to prove Lemma 2.3 for each graph \(G\) such that \(G[X] \) is connected. For this, we give a proof by induction on the cardinality of \(X\). If \(|X| = 1\), then, by Claim 3.1, Lemma 2.3 is true. We now suppose that the result is true for every \(T\)-graph \(T(u, v, X, Y)\) such that \(|X| < p\) (induction hypothesis), with \(p \geq 2\). Let us consider a \(T\)-graph \(G = T(u, v, X, Y)\), satisfying conditions (i) and (ii), with \(|X| = p\). If \(X\) is a stable set, then we are done (by Claim 3.1). So, in the following, we suppose that \(X\) is not a stable set.

We denote by \(y_1, \ldots, y_{2k}\) the vertices of \(Y\) such that \([u, y_1, \ldots, y_{2k}, v]\) is a path from \(u\) to \(v\) in \(G\).

Let us choose, in \(G[X]\), a longest chordless path \([x_1, \ldots, x_l]\) \((l \geq 2)\). Since \(G[X]\) is not a clique and the path is maximum, we have \(l \geq 3\). We can also remark that the two graphs \(G[X \setminus \{x_1\}]\) and \(G[X \setminus \{x_l\}]\) are connected. Moreover, it is easy to see that:

**Remark 3.2.** If \(G[X \setminus \{x_1, x_l\}]\) is not connected, then a vertex \(x \in X \setminus \{x_1, \ldots, x_l\}\) exists such that \(G[x_1, \ldots, x_l, x]\) is a chordless cycle.

We now consider the following sets of vertices:

\[Y_1 = \{ y \in Y \mid X \setminus \{x_1\} \subseteq N(y) \} \quad \text{and} \quad Y_2 = \{ y \in Y \mid X \setminus \{x_l\} \subseteq N(y) \}.\]

Clearly, the \(T\)-graphs \(T(u, v, X \setminus \{x_1\}, Y)\) and \(T(u, v, X \setminus \{x_l\}, Y)\) satisfy conditions (i) and (ii). Then, applying the induction hypothesis, we obtain that \(Y_1\) and \(Y_2\) are nonempty sets. If \(Y_1\) and \(Y_2\) have a common vertex \(y\), then the proof is finished (\(y\) is adjacent to every vertex of \(X\)).

Now, we assume that \(Y_1\) is disjoint from \(Y_2\).

If \(|Y| = 2\), then we can suppose, without loss of generality, that \(Y_1 = \{y_1\}\) and \(Y_2 = \{y_2\}\). Since \(G\) is a graph satisfying condition (i), the graph \(G[\{y_1, x_1, \ldots, x_l, y_2\}]\) is an even chordless path and \(G[\{y_1, y_2, x_1, \ldots, x_l, u, v\}]\) is an odd antihole, a contradiction. Thus, we have \(|Y| \geq 4\).
CLAIM 3.2. The chordless path $G[\{x_1, \ldots, x_l\}]$ has an odd length.

Proof. Assume the contrary. Then $G[\{x_1, \ldots, x_l\}]$ is an even chordless path. Let $y_i$ be a vertex of $Y_1$ and $y_j$ be a vertex of $Y_2$. We can suppose, without loss of generality, that $i < j$. Since $G[\{x_1, \ldots, x_l, y_i, y_j\}]$ is not an odd antihole, $y_i$ is adjacent to $y_j$. Moreover, if $|Y| \geq 4$ implies that one of the two vertices $u, v$ is nonadjacent to both vertices $y_i$ and $y_j$. Suppose, without loss of generality, that $u$ is nonadjacent to $y_i$ and nonadjacent to $y_j$ (then $y_i \neq y_j$). Since $[u, y_1, \ldots, y_2k, v]$ is an odd path, we can distinguish two cases: either $[u, y_1, \ldots, y_j]$ has an odd length or $[y_i, \ldots, y_{2k}, v]$ has an odd length. In the first case, we apply the induction hypothesis to the $T$-graph $T(u, y_1, X \setminus \{x_1\}, \{y_1, \ldots, y_{i-1}\})$ (which veriﬁes conditions (i) and (ii)), to deduce that there exists a vertex $y$ of $Y_1$ belonging to $\{y_1, \ldots, y_{i-1}\}$. Then $y$ is nonadjacent to $y_i$ and $G[\{x_1, \ldots, x_l, y, y_j\}]$ is an odd antihole, a contradiction. In the second case, the two chordless paths $[y_i, y_j, \ldots, y_{2k}, v]$ and $[u, y_1, \ldots, y_i, y_j]$ have an odd length. By applying the induction hypothesis to the $T$-graphs $T(y_i, v, X \setminus \{x_1\}, \{y_i, \ldots, y_{2k}\})$ and $T(u, y_1, X \setminus \{x_1\}, \{y_1, \ldots, y_{i-1}\})$ (which veriﬁes conditions (i) and (ii)), we deduce that there exists a vertex $y$ of $Y_1$ belonging to $\{y_i, \ldots, y_{2k}\}$ and there exists a vertex $y'$ of $Y_2$ belonging to $\{y_1, \ldots, y_j\}$. Since $y$ is nonadjacent to $y'$ (because $y \neq y_j$ and $y' \neq y_i$), the graph $G[\{x_1, \ldots, x_l, y, y'\}]$ is an odd antihole, a contradiction.  

If $G(X \setminus \{x_1, x_l\}]$ is nonconnected, then, by Remark 3.2, a vertex $x \in X$ exists such that $G[\{x_1, \ldots, x_l, x\}]$ is a chordless cycle. Since $l \geq 3$ and $G$ is a Berge graph, $G[\{x_1, \ldots, x_l\}]$ is an even chordless path, which is in contradiction with Claim 3.2. Hence, in the following, we can suppose that $G(X \setminus \{x_1, x_l\}]$ is connected.

Now, we call a special interval any chordless path $[y_n, \ldots, y_{n_0}]$ in $G[Y]$ such that $i_0 < l$, $y_i$ belongs to $Y_1$ (resp. $y_{i_0} \in Y_2$), $y_{i_0}$ belongs to $Y_2$ (resp. $y_n \in Y_1$), and none of the internal vertices belongs to $Y_1 \cup Y_2$. Obviously, $Y$ contains at least one special interval. We now give some properties on special intervals.

CLAIM 3.3. If $I = [y_n, \ldots, y_{n_0}]$ is a special interval in $G[Y]$, then we have $y_n, y_{n_0} \notin E$, $[y_n, \ldots, y_{n_0}]$ is an even path, and $\{y_n, y_{n_0}\} \cap \{y_i, y_{2k}\} = \emptyset$.

Proof. Let $[y_n, \ldots, y_{n_0}]$ be a special interval. We can suppose, without loss of generality, that $y_n \in Y_1$ and $y_{n_0} \in Y_2$.

Assume that $y_n, y_{n_0} \in E$. Then one of the vertices $u, v$ is nonadjacent to $y_n$ and nonadjacent to $y_{n_0}$. We suppose, without loss of generality, that $u$ is nonadjacent to both $y_n$ and $y_{n_0}$ (then $y_n \neq y_{n_0}$). Since $G[\{x_1, \ldots, x_l, y_n, y_{n_0}, u\}]$ is not an odd antihole, the graph $G[\{x_1, \ldots, x_l\}]$ is an even chordless path, which is in contradiction with Claim 3.2. Hence, we have $y_n, y_{n_0} \notin E$. 

Now, suppose that \( \{ y_i, \ldots, y_{2l} \} \) is an odd chordless path (greater than one). We apply the induction hypothesis to the \( T \)-graph \( T(y_i, y_{2l}, X \setminus \{x_1, x_2\}, \{y_{i+1}, \ldots, y_{2l-1}\}) \) (which verifies the conditions (i) and (ii)), and we obtain that there exists in \( \{ y_{i+1}, \ldots, y_{2l-1}\} \), a vertex \( y \) which is adjacent to all the vertices of \( X \setminus \{x_1, x_2\} \). According to the choice of the vertices \( y_i \) and \( y_{2l} \), the vertex \( y \) does not belong to \( Y_1 \cup Y_2 \), so \( y \) is nonadjacent to both \( x_1 \) and \( x_2 \). Then, \( G(T(x_1, \ldots, x_n, y)) \) is a chordless cycle. Since \( G \) is a Berge graph, \( G(T(x_1, \ldots, x_n, y)) \) is an even chordless path, which is incompatible with Claim 3.2. Hence, the path \( \{ y_i, \ldots, y_{2l} \} \) has an even length.

Assume that \( \{ y_i, y_{2l} \} \cap \{ y_1, y_{2k} \} \neq \emptyset \). Then, we can suppose, without loss of generality, that \( y_i = y_1 \) and we can apply the induction hypothesis to the \( T \)-graph \( T(y_1, y_{2l}, X \setminus \{x_1, x_2\}, \{y_{i+1}, \ldots, y_{2l-1}\}) \) (which verifies the conditions (i) and (ii)). Thus, there exists a vertex of \( Y_2 \) in \( \{ y_{i+1}, \ldots, y_{2l-1}\} \) which contradicts the fact that \( \{ y_i, \ldots, y_{2l}\} \) is a special interval. Hence, we have \( \{ y_i, y_{2l} \} \cap \{ y_1, y_{2k} \} = \emptyset \).

Claim 3.4. Let \( I = \{ y_i, \ldots, y_{2l}\} \) be a special interval in \( G(Y) \). If \( I' = \{ y_i, \ldots, y_{2l}\} \) is a special interval in \( G(Y) \) disjoint from \( I \) (i.e., \( \{ y_i, \ldots, y_{2l}\} \cap \{ y_i, \ldots, y_{2l}\} = \emptyset \) which is the closest to \( I \), then, in \( G(Y) \), each of the chordless paths between a vertex of \( \{ y_{i}, y_{2l}\} \) and a vertex of \( \{ y_{i}, y_{2l}\} \) has an even length.

Proof. Recall that we have \( j_1 < j_2 \) and \( i_1 < i_2 \). We can suppose, without loss of generality, that \( j_2 < i_1 \), \( y_i \in Y_1 \) and \( y_{2l} \in Y_2 \). By Claim 3.3, we know that \( I \) and \( I' \) both have an even length. If the chordless path \( P = \{ y_{i}, \ldots, y_{2l}\} \) has an even length, then the claim is proved. So, we assume that \( P \) has an odd length.

If \( y_i \in Y_2 \) (resp. \( y_{2l} \in Y_1 \)), then we apply the induction hypothesis to the \( T \)-graph \( T(y_i, y_{2l}, X \setminus \{x_1, x_2\}, \{y_{i+1}, \ldots, y_{2l-1}\}) \) (resp. \( T(y_i, y_{2l}, X \setminus \{x_1, x_2\}, \{y_{i+1}, \ldots, y_{2l-1}\}) \)), which verifies the conditions (i) and (ii), and we deduce that there exists a vertex \( y \) in \( \{ y_{i+1}, \ldots, y_{2l-1}\} \cap Y_2 \) (resp. in \( \{ y_{i+1}, \ldots, y_{2l-1}\} \cap Y_1 \)). Since \( I \) and \( I' \) are special intervals, \( y \) cannot belong to \( \{ y_{i+1}, \ldots, y_{2l-1}\} \) (resp. to \( \{ y_{i+1}, \ldots, y_{2l-1}\} \cup \{ y_{i+1}, \ldots, y_{2l-1}\} \)). Then \( y \) belongs to \( \{ y_{i+1}, \ldots, y_{2l-1}\} \) (so \( P \) contains at least three edges). Moreover, we can apply the induction hypothesis to the \( T \)-graph \( T(y_i, y_{2l}, X \setminus \{x_1, x_2\}, \{y_{i+1}, \ldots, y_{2l-1}\}) \) (resp. \( T(y_i, y_{2l}, X \setminus \{x_1, x_2\}, \{y_{i+1}, \ldots, y_{2l-1}\}) \)), which verifies the conditions (i) and (ii), and we deduce that there exists a vertex \( y' \) in \( \{ y_{i+1}, \ldots, y_{2l-1}\} \cap Y_1 \). Hence, there exists a special interval \( I'' \) in \( \{ y_{i+1}, \ldots, y_{2l-1}\} \), which is incompatible with the fact that the special interval \( I'' \) is the closest to \( I \).

To finish the proof of Lemma 2.3, let \( I = \{ y_i, \ldots, y_{2l}\} \) be a special interval. We can suppose without loss of generality that \( y_i \in Y_1 \) and \( y_{2l} \in Y_2 \). Since \( I \) is an even chordless path (by Claim 3.3) and \( \{ y_i, \ldots, y_{2l}, v \} \) has
Moreover, since all the chordless paths between $x$ and $y$ have an odd length (by hypothesis), one of the two paths $[y_1, \ldots, y_{n-1}]$ and $[y_{n+1}, \ldots, y_{2k+1}]$ has an odd length (we know, by Claim 3.3, that $y_1$, $y_2$ and $y_{2k}$ are odd). We can suppose, without loss of generality, that $[y_1, \ldots, y_{n-1}]$ has an odd length and that $I$ is as close as possible to $u$, with this property. Now, we can apply the induction hypothesis to the $T$-graph $T[u, y_{n}, X \setminus \{x_i\}]$ (resp. $T[u, y_{n}, X \setminus \{x_{i-1}\}]$), which verifies the conditions (i) and (ii), and we deduce that there exists a vertex of $Y_1$ (resp. of $Y_2$) in $[y_1, \ldots, y_{n-1}]$ (resp. in $[y_1, \ldots, y_{n-1}]$) because $[y_1, \ldots, y_{n-1}]$ contains no vertex of $Y_2$. Then there exists special intervals in $[y_1, \ldots, y_{n-1}]$. Among these special intervals, we consider the interval $I'=[y_{j_1}, \ldots, y_{j_2}]$ (with $j_1 < j_2$) which is the closest to $I$. By Claim 3.4, the chordless path $[y_{j_1}, \ldots, y_{j_2}]$ has an even length. Thus the chordless path $[y_1, \ldots, y_{n-1}]$ is odd, which is in contradiction with the choice of $I$. So, Lemma 2.3 is proved.

We now give notions used in the proof of Lemma 2.2.

Let $G=(V, E)$ be a connected Berge graph with a cutset $S$. In the following, we will denote by $C$ a connected component of $G[V \setminus S]$ and $D$ the set $V \setminus (S \cup C)$. If $G$ admits a skew partition $(A, B, C, D)$, without loss of generality, we will suppose that $C$ is a connected component of $G[V \setminus (A \cup B)]$ and $D=V \setminus (A \cup B \cup C)$.

Let $X$ be a subset of $V$. Let $u, v$ be two nonadjacent vertices of $G$. We say that $[u, v]$ is an $X$-even pair (resp. an $X$-odd pair) if each chordless path between $u$ and $v$ in $G[X \cup \{u, v\}]$ has an even length (resp. odd length). Clearly, every even pair of $G$ is a $V$-even pair. Moreover, we can observe that if there exists no chordless path between $u$ and $v$ in $G[X \cup \{u, v\}]$, then $[u, v]$ is both an $X$-even pair and an $X$-odd pair. We say that $[u, v]$ is a connected $X$-even pair (resp. a connected $X$-odd pair) if $[u, v]$ is an $X$-even pair (resp. an $X$-odd pair) and if there exists at least one even (resp. odd) chordless path from $u$ to $v$ in $G[X \cup \{u, v\}]$. Now, we have:

**Lemma 3.1.** Let $G$ be a Berge graph which admits a skew partition $(A, B, C, D)$. Let $u, v$ be two nonadjacent vertices of $G[A]$ and let $x, y$ be two nonadjacent vertices of $G[B]$ such that $[u, v]$ and $[x, y]$ are two connected $(C \cup D)$-odd pairs. Then, $G[u, v, z_1, \ldots, z_{2k}]$ is a chordless path from $u$ to $v$ in $G[u, v] \cup C \cup D$ if and only if $G[[x, y, z_1, \ldots, z_{2k}]]$ is a chordless path from $x$ to $y$ in $G[[x, y] \cup C \cup D]$.

**Proof.** It is sufficient to prove the “only if” part.

Let $P=[u, z_1, \ldots, z_{2k}, v]$ be an odd chordless path from $u$ to $v$ in $G[u, v] \cup C \cup D$. Recall that $uu, xx, yy, uv \in E$. Since $G$ contains no odd hole, both the vertices $x$ and $y$ are adjacent to some vertices of $[z_1, \ldots, z_{2k}]$.

Moreover, since all the chordless paths between $x$ and $y$ in $G[[x, y] \cup C \cup D]$ are odd paths, $x$ and $y$ cannot have a common neighbour in
\{z_1, \ldots, z_{2k}\}. Let \(z_i\) be a neighbour of \(x\) in \([z_1, \ldots, z_{2k}]\) and \(z_j\) be a neighbour of \(y\) in \([z_1, \ldots, z_{2k}]\) such that the chordless subpath joining \(z_i\) to \(z_j\) in \([z_1, \ldots, z_{2k}]\) has the minimum length. We can suppose, without loss of generality, that \(i < j\). Hence \([z_i, \ldots, z_j]\) is an odd subpath of \(P\) and \(x\) (resp. \(y\)) has no neighbour in \([z_{i+1}, \ldots, z_{j-1}]\). Since \(G[[u, x, y, z_i, \ldots, z_j]]\) (resp. \(G[[v, x, y, z_i, \ldots, z_j]]\)) is not an odd hole, \(u\) (resp. \(v\)) has a neighbour in \([z_i, \ldots, z_j]\). Now, it is not difficult to see that the only one possibility is that \(z_i = z_1\) and \(z_j = z_{2k}\). Thus, \(G[[x, y, z_1, \ldots, z_{2k}]]\) is a chordless path between \(x\) and \(y\). \(\square\)

Considering the skew partition \((C, D, A, B)\) of the graph \(\bar{G}\), we obtain the following corollary:

**Corollary 3.1.** Let \(u', v'\) be two adjacent vertices of \(G[C]\) and let \(x', y'\) be two adjacent vertices of \(G[D]\) such that, in \(G\), \(\{u', v'\}\) and \(\{x', y'\}\) are two connected \((A \cup B)\)-odd pairs. Then, \(G[[u', v', z_1, \ldots, z_{2k}]]\) is a chordless path from \(u'\) to \(v'\) in \(G[[u', v', \emptyset, A \cup B]]\) if and only if \(G[[x', y', z_1, \ldots, z_{2k}]]\) is a chordless path from \(x'\) to \(y'\) in \(G[[x', y', \emptyset, A \cup B]]\).

**Proof of Lemma 2.2.** Let \(G = (V, E)\) be a minimal imperfect graph. By Theorem 1.1, the graph \(\bar{G}\) is also a minimal imperfect graph. By contradiction, assume that \(G\) admits a skew partition \((A, B, C, D)\) with \(|A| = 2\). We denote by \(u\) and \(v\) the two vertices of \(A\). Since odd holes and odd antiholes have no double star cutset, we can assume that \(G\) is a Berge graph. Recall that \(C\) is a connected component of \(G[V \setminus (A \cup B)]\) and \(D = V \setminus (A \cup B \cup C)\). We shall prove that \(C\) and \(D\) are connected components of \(G[V \setminus (A \cup B)]\) and that all of the chordless paths from \(u\) to \(v\) in \(G[[u, v] \cup C \cup D]\) have length three. In the end, to derive a contradiction, we shall construct a \(T\)-cutset of \(G\). We first explain some remarks:

**Remark 3.3.** Vertices \(u\) and \(v\) are not adjacent; otherwise, \(A \cup B\) is a star cutset of \(G\), which is in contradiction with Lemma 1.1.

**Remark 3.4.** The set \(B\) is not a clique of \(G\); otherwise, \(A \cup B\) is a star cutset of \(G\).

**Remark 3.5.** Both vertices \(u\) and \(v\) have neighbours in \(C\) and in \(D\); otherwise, \([u] \cup B\) or \([v] \cup B\) is a star cutset of \(G\), a contradiction. So, there exist chordless paths between \(u\) and \(v\) in \(G[[u, v] \cup C]\) and in \(G[[u, v] \cup D]\). Moreover, by Remark 2.2, all these paths have the same parity.

**Remark 3.6.** Every chordless path between \(u\) and \(v\) in \(G[[u, v] \cup C \cup D]\) is an odd path; otherwise, by Remark 3.5, \(u, v\) is an even pair in
G and by Lemma 2.1, G cannot be a minimal imperfect graph. Consequently, every vertex of B has neighbours in C and in D (otherwise, G contains an odd hole).

By Claims 3.5 and 3.6 below, \( \{u, v\} \) is a connected \((C \cup D)\)-odd pair. Then \( u \) and \( v \) have no common neighbour in \( C \cup D \).

**Claim 3.5 [5]**. The graph \( G[D] \) is connected. Moreover, we can suppose, without loss of generality, that in every \( \omega(G) \)-colouring of \( G[A \cup B \cup C] \), vertices \( u \) and \( v \) receive different colours, and in every \( \omega(G) \)-colouring of \( G[A \cup B \cup D] \), vertices \( u \) and \( v \) receive the same colour.

**Proof.** We will prove that \( G[A \cup B \cup Q_i] \) is connected. Moreover, we can suppose, without loss of generality, that in every \( \omega(G) \)-colouring of \( G[A \cup B \cup Q_i] \), vertices \( u \) and \( v \) receive different colours; and in every \( \omega(G) \)-colouring of \( G[A \cup B \cup Q_j] \), vertices \( u \) and \( v \) receive the same colour.

Otherwise, one of the following situations holds:

(situation 1) for every \( i, 1 \leq i \leq p \), there is a \( \omega(G) \)-colouring \( C_i \) of \( G[A \cup B \cup Q_i] \) such that \( u \) and \( v \) receive different colours;

(situation 2) for every \( i, 1 \leq i \leq p \) there is a \( \omega(G) \)-colouring \( C_i \) of \( G[A \cup B \cup Q_i] \) such that \( u \) and \( v \) receive the same colour.

Let us denote by \( S_i \) the colour of \( C_i \) containing \( u \). For every \( i, 1 \leq i \leq p \), we have, in the first situation, \( S_i \cap (A \cup B) = \{u\} \), and in the second situation, \( S_i \cap (A \cup B) = \{u, v\} \). Then, the set \( S = \bigcup_{i=1}^{p} S_i \) is a stable set which intersects all the \( \omega \)-cliques of \( G \). Hence, we have \( \omega(G[V \setminus S]) = \omega(G) - 1 \), and since \( G[V \setminus S] \) is perfect, there is a colouring of \( G[V \setminus S] \) in \( \omega(G) - 1 \) colours. Therefore, \( G \) is colourable in \( \omega(G) \) colours, a contradiction.

Now, we suppose that \( p \geq 3 \). Then, let us consider a component \( Q_k \) different from \( Q_i \) and \( Q_j \) and let us choose a vertex \( x_k \) in \( Q_k \). The \( \omega(G) \)-colouring \( C \) of \( G - x_k \) induces an \( \omega(G) \)-colouring of \( G[A \cup B \cup Q_k] \) and an \( \omega(G) \)-colouring of \( G[A \cup B \cup Q_j] \). If in the colouring \( C \), \( u \) and \( v \) receive the same colour (resp. receive different colours), then the condition (a) (resp. (b)) is contradicted. Thus, \( p = 2 \), and without loss of generality, we can suppose that \( C = Q_i \) and \( D = Q_j \).

Since the graphs \( G[C] \) and \( G[D] \) are connected, by Remarks 3.5 and 3.6, the set \( A \cup B \) is a minimal cutset of \( G \). We can now remark:
Remark 3.7. Let $c$ be a vertex of $C$ and let $\mathcal{C}$ be the $\omega(G)$-colouring of $G - c$. By Claim 3.5, vertices $u$ and $v$ receive the same colour in the colouring $\mathcal{C}$. Then, $u$ and $v$ belong to a common $\omega(G)$-stable of $G$. Thus, the set $C \cup D \setminus (N(u) \cup N(v))$ is not empty; otherwise $\omega(G) = 2$ which is incompatible with the fact that $G$ is a minimal imperfect Berge graph.

We obtain the following claims:

**Claim 3.6.** For each neighbour $a$ of $u$ (resp. $v$) in $C \cup D$, there exists a chordless path $P$ joining $u$ to $v$ in $G[\{u, v\} \cup C \cup D]$ such that $u$ (resp. $v$) and $a$ are consecutive on $P$.

**Proof.** Clearly, it is sufficient to show that the result holds for any vertex $a$ in $N(u) \cap C$ (in other cases, the proof is similar). Since $\{u\} \cup N(u)$, $\{u\}$ is not a star cutset of $G$, there exists in $G[\{v, a\} \cup (C \setminus N(u))]$ a chordless path, $[a, z_1, ..., z_k, v]$, from $a$ to $v$. Moreover since no internal vertex of this path is adjacent to $u$, $[u, a, z_1, ..., z_k, v]$ is a chordless path joining $u$ to $v$ such that $u$ and $a$ are consecutive on this path.

**Claim 3.7.** Let $x$ and $y$ be two nonadjacent vertices in $G[B]$. Then $\{x, y\}$ is a connected $(C \cup D)$-even pair.

**Proof.** By the end of Remark 3.6, there exist some chordless paths joining $x$ to $y$ in both $G[\{x, y\} \cup C]$ and $G[\{x, y\} \cup D]$. Then, by Remark 2.2, $\{x, y\}$ is either a connected $(C \cup D)$-even pair or a connected $(C \cup D)$-odd pair. If $\{x, y\}$ is a connected $(C \cup D)$-even pair, then the proof is finished. Now, we suppose that $\{x, y\}$ is a connected $(C \cup D)$-odd pair. Then, $x$ and $y$ have no common neighbour in $C \cup D$.

**Fact 1.** Every neighbour of $u$ (resp. $v$) in $C \cup D$ is adjacent to one of the vertices $x$ and $y$.

Let $a$ be a neighbour of $u$ in $C \cup D$ (in the case where $a$ is a neighbour of $v$, the proof is similar). By Claim 3.6, there exists a chordless path $[u, a, z_1, ..., z_k, v]$ in $C \cup D \cup \{u, v\}$. By Lemma 3.1, $[x, a, z_1, ..., z_k, y]$ or $[y, a, z_1, ..., z_k, x]$ is a chordless path from $x$ to $y$. Then, the vertex $a$ is a neighbour of $u$ or $v$.

**Fact 2.** We have either $N(u) \cap C \subseteq N(x)$ and $N(v) \cap C \subseteq N(y)$ or $N(v) \cap C \subseteq N(x)$ and $N(u) \cap C \subseteq N(y)$.

By Fact 1, we have $N(u) \cap C \subseteq N(x) \cup N(y)$ and $N(v) \cap C \subseteq N(x) \cup N(y)$. Assume that Fact 2 is false; without loss of generality, we can suppose that $N(v) \cap C$ intersects bot of the sets $N(x)$ and $N(y)$. The, let us consider a minimum chordless path, denoted by $P$, joining a vertex of
$N(v) \cap N(x)$ to a vertex of $N(v) \cap N(y)$ in $G[C]$. Let $a$ and $b$ be the end-vertices of $P$ respectively in $N(v) \cap N(x)$ and $N(v) \cap N(y)$.

Since $u$ and $v$ have no common neighbor in $C \cup D$, we have $uv \notin E$ and $uv \notin E$. If $a$ is adjacent to $b$ then $\{u, x, a, b, y\}$ induces an odd hole of $G$, a contradiction. Therefore $a$ is not adjacent to $b$. Let us set $P=[a, z_1, \ldots, z_l, b]$. Since this path has a minimum length, we can observe that none of the vertices $z_1, \ldots, z_l$ belongs to $(N(v)\cap N(x))\cup (N(v)\cap N(y))$. By Fact 1, the vertex $v$ is adjacent to none of the vertices $z_1, \ldots, z_l$. Clearly, there exists an odd chordless path joining $x$ to $y$ in $G[\{a, z_1, \ldots, z_l, b\} \cup \{x, y\}]$. By Lemma 3.1, $u$ has a neighbour in $\{z_1, \ldots, z_l\}$. Moreover, the neighbours of $u$ in $\{z_1, \ldots, z_l\}$ are all adjacent to $x$ or all adjacent to $y$; otherwise, there exists a chordless path from $x$ to $y$ in $G[\{z_1, \ldots, z_l\}]$ and, by Lemma 3.1, $v$ is adjacent to a vertex of $\{z_1, \ldots, z_l\}$, a contradiction. We can suppose, without loss of generality, that they are adjacent to $x$. Now, let us choose in $\{a, z_1, \ldots, z_l\}$, the neighbour of $u$, noted $z_1$, which is the closest to $a$. The path $[v, a, z_1, \ldots, z_l, u]$ is a chordless path from $v$ to $u$, and by Lemma 3.1, one of the vertices $z_1$ and $a$ is adjacent to $y$, a contradiction.

Henceforth, we can suppose, without loss of generality, that we have $N(u) \cap C \subseteq N(x)$ and $N(v) \cap C \subseteq N(y)$.

**Fact 3.** We have $N(x) \cap C \subseteq N(u)$ and $N(y) \cap C \subseteq N(v)$.

We prove that $N(x) \cap C \subseteq N(u)$ (the proof is similar when $N(y) \cap C \subseteq N(v)$). In the contrary case, there exists a vertex $z$ in $C$ which is adjacent to $x$ and nonadjacent to $u$. Since $\{u\} \cup N(u)$ is not a star cutset of $G$, there exists in $G[C \setminus N(u)] \cup \{v\}$ a chordless path $[z, z_1, \ldots, z_k, v]$ from $z$ to $v$ ($k \geq 1$). Since $yz_k \in E$ and $xz \in E$, there exists in $G[\{x, z, z_1, \ldots, z_k, y\}]$ a chordless path from $x$ to $y$. Then, by Lemma 3.1, one of the vertices $z$, $z_1, \ldots, z_k$ is adjacent to $u$, a contradiction.

By Fact 3, we have $N(u) \cap C = N(z) \cap C$ and $N(v) \cap C = N(z) \cap C$. By a similar proof of Facts 2 and 3, we can prove that either $N(u) \cap D = N(y) \cap D$ and $N(v) \cap D = N(x) \cap D$ or $N(u) \cap D = N(x) \cap D$ and $N(v) \cap D = N(y) \cap D$. Thus, we have $(N(u) \cup N(v)) \cap (C \cup D) = (N(x) \cup N(y)) \cap (C \cup D)$.

Now, let $S'$ be an $\alpha$-stable of $G$, containing vertices $u$ and $v$. The set $S' = (S' \setminus \{u, v\}) \cup \{x, y\}$ is an other $\alpha$-stable of $G$. By Property (P2), the clique $K(S)$ intersects $S'$ and we can suppose, without loss of generality, that $x$ belongs to $K(S)$. Moreover, $K(S)$ is an $\omega$-clique of $G[A \cup B \cup C]$. Otherwise, let $z$ be a vertex in $K(S) \cap D$. The $\omega(G)$-colouring of $G-\{z\}$ induces an $\omega(G)$-colouring of $G[A \cup B \cup C]$, denoted by $\mathcal{C}$. By Property (P3), $S \cap (A \cup B \cup C)$ is a colour of $\mathcal{C}$, which contradicts Claim 3.5. Since $N(u) \cap C = (N(x) \cap C$, the set $K(S) \cup \{u\}$ is an $(\omega+1)$-clique of $G$, a contradiction. So, Claim 3.7 is proved.
Let us now consider a connected component of $\overline{G}[B]$, denoted by $Q$. Observe that $(Q, A \cup B, Q, C, D)$ is a skew partition of $G$. To finish the proof of Lemma 2.2, we will prove that $G$ or $\overline{G}$ has a $T$-cutset, which is in contradiction with the fact that $G$ is a minimal imperfect graph (by Lemma 1.4).

Claim 3.8. Each of the chordless paths between $u$ and $v$ in $G[\{u,v\} \cup C \cup D]$ has length three. Moreover, for every path $[u,z_1,z_2,v]$ in $G[\{u,v\} \cup C \cup D]$, $\{z_1,z_2\}$ is a connected $(A \cup B)$-odd pair in $G$.

Proof. Since $G$ admits no $T$-cutset, either each vertex of $C$ is not adjacent to all the vertices of $Q$, or each vertex of $D$ is not adjacent to all the vertices of $Q$. We can suppose, without loss of generality, that there exists no vertex in $C$ which is adjacent to all the vertices of $Q$. Let us consider an odd chordless path $P=[u,s_1,\ldots,s_{2k},v]$, between $u$ and $v$ in $G[\{u,v\} \cup C]$. According to Claim 3.7, the $T$-graph $H=T(u,v,\overline{G},\{s_1,\ldots,s_{2k}\})$ satisfies condition (ii) of Lemma 2.3. Since none of the vertices $s_1,\ldots,s_{2k}$ is adjacent to all of the vertices of $Q$, the graph $H$ does not satisfy the condition (i) of Lemma 2.3. So, we have $P=[u,s_1,s_2,v]$ (by Remark 2.1), and there exists an odd chordless path $[s_1,q_1,\ldots,q_{2l},s_2]$ between $s_1$ and $s_2$ in $G[\{s_1,s_2\} \cup Q]$. Now, by Remark 3, all the paths between $s_1$ and $s_2$ in both of the graphs $G[\{s_1,s_2\} \cup A]$ and $G[\{s_1,s_2\} \cup B]$ have an odd length. Then, in $G$, $\{s_1,s_2\}$ is connected $(A \cup B)$-odd pair.

Moreover, no vertex $d$ in $D$ is adjacent to all the vertices $q_1,\ldots,q_{2l}$ (otherwise, $G[\{d,s_1,s_2,q_1,\ldots,q_{2l}\}]$ is an odd antihole). Then, no vertex in $D$ is adjacent to all the vertices of $Q$. Thus, by using the same arguments as above, we can assert that all the chordless paths between $u$ and $v$ in $G[\{u,v\} \cup D]$ have length three, and if $[u,d_1,d_2,v]$ is such a path, then $\{d_1,d_2\}$ is a connected $(A \cup B)$-odd pair in $G$.

Claim 3.9. Let $z_1,z_2$ be vertices of $C \cup D$ such that $[u,z_1,z_2,v]$ is a chordless path in $G$. Let $P=[z_1,q_1,\ldots,q_l,z_2]$ be a chordless path between $z_1$ and $z_2$ in $G[\{z_1,z_2\} \cup Q]$. Then $P$ has an odd length, $l \geq 4$, and for every path $[u,s_1,s_2,v]$ in $G[\{u,v\} \cup C \cup D]$, the graph $G[\{s_1,s_2,q_1,\ldots,q_{2l}\}]$ is an odd chordless path from $s_1$ to $s_2$.

Proof. We can suppose, without loss of generality, that $z_1$ and $z_2$ belong to $C$. Since $P+u+v$ cannot be an odd antihole, $P$ is an odd chordless path in $G$. If $l=2$, then $[q_1,2,z_1,q_2]$ is an odd chordless path between $q_1$ and $q_2$ in $G$, which is in contradiction with Claim 3.7. Then, we have $l \geq 4$.

Let $d_1,d_2$ be vertices of $D$ such that $[u,d_1,d_2,v]$ is a chordless path in $G$. Recall that, in $G$, $\{d_1,d_2\}$ and $\{z_1,z_2\}$ are connected $(A \cup B)$-odd
pairs (by Claim 3.8). Then, by Corollary 3.1, \( \bar{G}[\{d_1, d_2, q_1, \ldots, q_l\}] \) is an odd chordless path from \( d_1 \) to \( d_2 \). Let \( c_1, c_2 \) be vertices of \( G \) such that \([u, c_1, c_2, v]\) is a chordless path in \( G \). Again, by Claim 3.8 and Corollary 3.1, \( \bar{G}[\{c_1, c_2, q_1, \ldots, q_l\}] \) is an odd chordless path from \( c_1 \) to \( c_2 \) (considering, in \( \bar{G} \), the two connected \( (A \cup B) \)-odd pairs \( \{d_1, d_2\} \) and \( \{c_1, c_2\} \)).

Let \( c_1, c_2 \) be vertices of \( C \) such that \([u, c_1, c_2, v]\) is a chordless path in \( G \). By Claim 3.8, there exists an odd chordless path between \( c_1 \) and \( c_2 \) in \( \bar{G}[Q \cup \{c_1, c_2\}] \). Moreover, by Claim 3.9, each chordless path joining \( c_1 \) to \( c_2 \) in \( \bar{G}[Q \cup \{c_1, c_2\}] \) has an odd length.

Let us denote by \( Q' \) the set of all the vertices \( q_1 \) from \( N_G(c_1) \cap Q \) such that \( q_1 \) belongs to a path joining \( c_1 \) to \( c_2 \) in \( \bar{G}[Q \cup \{c_1, c_2\}] \) (then, on such a path, \( q_1 \) and \( c_1 \) are consecutive). In the same way, we denote by \( Q'' \) the set of all the vertices \( q_2 \) from \( N_G(c_2) \cap Q \) such that \( q_2 \) belongs to a path joining \( c_1 \) to \( c_2 \) in \( \bar{G}[Q \cup \{c_1, c_2\}] \). Now, set \( Q_1 = Q' \cup Q'' \). We obtain the following result:

**Claim 3.10.** There exists no edge between \( Q_1 \) and \( (C \cup D) \setminus (N(u) \cup N(v)) \).

**Proof.** Let \( q_1 \) be a vertex of \( Q_1 \). Then, there exist vertices \( q_2, \ldots, q_{2l} \) of \( Q \) such that \([c_1, q_1, \ldots, q_{2l}, c_2]\) or \([c_2, q_1, \ldots, q_{2l}, c_1]\) is an odd chordless path from \( c_1 \) to \( c_2 \) in \( G \).

Let \( z \) be a vertex in \( D \setminus (N(u) \cup N(v)) \). Since \([u] \cup N(u)\) (resp. \([v] \cup N(v)\)) is not a star cutset of \( G \), there exists, in \( \bar{G}(D \setminus (N(u) \cup [v]))\) (resp. \( \bar{G}(D \setminus (N(v) \cup [u]))\)), a chordless path \( P = [v, s_1, \ldots, s_p, z] \) with \( p \geq 1 \) (resp. \( P' = [u, t_1, \ldots, t_q, z] \) with \( q \geq 1 \)) from \( v \) (resp. \( u \)) to \( z \). Vertices \( s_1 \) and \( t_1 \) are adjacent; otherwise, there exists a chordless path between \( u \) and \( v \) in \( \bar{G}[u, v] \cup D \) with a length greater than three, which is in contradiction with Claim 3.8. Then, by Claim 3.9, \( \bar{G}[s_1, t_1, q_1, \ldots, q_{2l}] \) is an odd chordless path from \( s_1 \) to \( t_1 \).

We consider, in \( \bar{G} \), the \( T \)-graph \( H = T(c_1, c_2, \{s_1, \ldots, s_p, z\}, \{q_1, \ldots, q_{2l}\}) \). This \( T \)-graph satisfies the condition (i) of Lemma 2.3 (by Claim 3.7) and the condition (ii) of Lemma 2.3 (otherwise \( H + u \) contains an odd hole, because in \( \bar{G} \), \( u \) is adjacent to all the vertices of \( \{s_1, \ldots, s_p, z\} \) and \( u \) is non-adjacent to every vertex of \( \{q_1, \ldots, q_{2l}\} \)). Thus, by Lemma 2.3, there exists a vertex \( q_j \in \{q_1, \ldots, q_{2l}\} \) which is adjacent, in \( \bar{G} \), to all the vertices \( s_1, \ldots, s_p, z \) (in particular, in \( \bar{G} \)), \( s_1 \) and \( z \) are nonadjacent to \( q_j \). Since \( \bar{G}[s_1, t_1, q_1, \ldots, q_{2l}] \) is a chordless path from \( s_1 \) to \( t_1 \), we have \( q_j \in \{q_1, q_{2l}\} \). We consider now, in \( \bar{G} \), the \( T \)-graph \( H' = T(c_1, c_2, \{t_1, \ldots, t_q, z\}, \{q_1, \ldots, q_{2l}\}) \).

As above, we obtain that there exists a vertex \( q_j \in \{q_1, q_{2l}\} \) which is
adjacent, in \( G \), to all of the vertices \( t_1, \ldots, t_r, z \) (in particular, in \( G, t_1 \) and \( z \) are nonadjacent to \( q_j \)). We have \( q_i \neq q_j \) (otherwise, \( G[\{q_j, v, v, s_1, t_1\}] \) is an odd hole). Therefore, \( z \) is nonadjacent to vertices \( q_1 \) and \( q_2 \) in \( G \).

Similarly, we can prove the result for a vertex \( z \) in \( C\setminus(N(u) \cup N(v)) \).

We now consider a particular chordless path \([c_1, q_1, \ldots, q_{2l}, c_2]\) in \( G[\{c_1, c_2\} \cup Q]\) (we recall that, by Claim 3.9, \( l \geq 2 \)). Let us denote by \( Q_2 \) the connected component of \( G[Q \setminus Q_1] \) which contains vertices \( q_{2l}, \ldots, q_{2l-1} \).

**Claim 3.11.** There exist all possible edges between \((C \cup D) \cap (N(u) \cup N(v))\) and \( Q_2 \).

**Proof.** Assume the contrary. Without loss of generality, we can suppose that a vertex \( c \in N(u) \cap C \) is nonadjacent to a vertex \( t \in Q_2 \). According to Claim 3.6 and Claim 3.8, there exists a vertex \( c' \in N(v) \cap C \) such that \([u, c, c', v]\) is a chordless path from \( u \) to \( v \) in \( G[\{u, v\} \cup C]\). By Claim 3.9, \( G[\{c, c', q_1, \ldots, q_{2l}\}] \) is an odd chordless path from \( c \) to \( c' \). So, in \( G, c \) and \( c' \) are adjacent to all of the vertices \( q_{2l}, \ldots, q_{2l-1} \), and \( c' \) is nonadjacent to \( q \in \{q_1, q_2\} \). Since \( Q_2 \cup \{q\} \) is connected in \( G \), there exists, in \( G[Q_2 \cup \{q\}] \), a chordless path from \( t \) to \( q \). Therefore, a chordless path (denoted by \( P \)) from \( c \) to \( c' \) exists in \( G[\{c, c'\} \cup Q_2 \cup \{q\}] \). Then, by Claim 3.9, \( G[V(P) \cup \{c, c'\} \cup \{c_1, c_2\}] \) is a chordless path joining \( c_1 \) to \( c_2 \) in \( G[Q \cup \{c_1, c_2\}] \). Thus, \( P \) contains two vertices of \( Q_1 \), a contradiction.

Observe that there exist all possible edges between \( Q_2 \) and \( \{u, v\} \cup (B \setminus Q_1) \). Therefore, in \( G \), the set \( X = ((C \cup D) \setminus (N(u) \cup N(v))) \cup Q_1 \) is a cutset of \( G \); indeed, this set disconnects \( Q_2 \) and \( D' \setminus X = (V \setminus X) \cup Q_2 \). Let \( A' \) be a connected component of \( G[(C \cup D) \setminus (N(u) \cup N(v))] \). By Remark 3.7 the set \( A' \) is not empty. By Claim 3.10, we can see that \((A', X \setminus A', Q_2, D')\) is a skew cutset of \( G \). Remark that \( Q_2 \) is a connected component of \( G[V \setminus X] \) and that \( \{u, v\} \subseteq D' \). Moreover, in \( G \), \( u \) (resp. \( v \)) is adjacent to all of the vertices of \( A' \).

Let us now consider, in \( G \), the T-graph \( J = T(q_1, q_{2l}, A', \{q_{2l}, q_{2l-1}\}) \). According to Claim 3.7, the graph \( J \) satisfies the condition (i) of Lemma 2.3. If \( J \) does not verify the condition (ii) of Lemma 2.3, then there exists, in \( G \), an odd chordless path, denoted by \( P = \{a_1, q_1, \ldots, a_n, a_2\} \), between two nonadjacent vertices of \( A' \). The internal vertices of \( P \) belong to \( \{q_{2l}, \ldots, q_{2l-1}\} \). Since, in \( G \), \( u \) is nonadjacent to every vertex of \( \{q_{2l}, q_{2l-1}\} \), we have that \( P + u \) is an odd hole a contradiction. Hence, the T-graph \( J \) verifies the condition (ii). We can apply Lemma 2.3, and we obtain that there exists in \( \{q_{2l}, \ldots, q_{2l-1}\} \) a vertex \( q_i \) which is adjacent, in \( G \), to all of the vertices of \( A' \). Therefore, \( A' \cup (X \setminus A') \) is a T-cutset of \( G \), a contradiction. So, Lemma 2.2 is proved.
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